No open exact-root cover for Vassiliev's quadratic family

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Abstract

Problem 2A in Vassiliev's 2015 problem list asks for the minimum number of open coefficient sets supporting continuous exact real-root selections for a six-parameter quadratic system. For the full coefficient space printed there, no such open cover exists, of any cardinality. Every neighborhood of the zero coefficients contains the complex square-root obstruction. We give an elementary proof and a translated family of obstructed coefficients. This is an explicit diagnosis of the literal formulation using classical topology and the source's own square-root slice. Historical firstness is not established. No result is claimed for a modified finite-genus question or the separate approximate-root problem.

1 The printed question and its answer

For $p = (a_1, a_2, a_0, b_1, b_2, b_0) \in \mathbb{R}^6$, put

$$F_p(x, y) = x^2 - y^2 + a_1 x + a_2 y + a_0, \tag{1}$$
$$G_p(x, y) = xy + b_1 x + b_2 y + b_0. \tag{2}$$

An admissible set is a Euclidean-open $U \subset \mathbb{R}^6$ with a continuous map $s : U \to \mathbb{R}^2$ satisfying $F_p(s(p)) = G_p(s(p)) = 0$ for every $p \in U$. Vassiliev's Problem 2A asks for a minimum number of such sets covering all of $\mathbb{R}^6$ [1, Section 2, p. 205]. Repeated-root coefficients are included. A selection chooses an actual point, irrespective of algebraic multiplicity.

Theorem 1. *No open neighborhood of $0 \in \mathbb{R}^6$ is admissible. Consequently, no admissible open cover of $\mathbb{R}^6$ exists, even if arbitrarily many members are allowed.*

Thus the admissible-cover class is empty. If a finite-cover number is extended by assigning $+\infty$ when no finite admissible cover exists, its value here is $+\infty$; this notation does not assert the existence of an infinite admissible cover.

2 An elementary local proof

Lemma 1. *For every $r > 0$, there is no continuous $S : D_r \to \mathbb{C}$, where $D_r = \{w \in \mathbb{C} : |w| < r\}$, satisfying $S(w)^2 = w$.*

Proof. Consider the continuous path

$$\gamma(t) = \begin{cases} (1 - t) + it, & 0 \le t \le 1, \\ (1 - t) + i(2 - t), & 1 \le t \le 2. \end{cases}$$

Its two formulas agree at $t = 1$, and

$$|\gamma(t)|^2 = \begin{cases} 2(t - \frac{1}{2})^2 + \frac{1}{2}, & 0 \le t \le 1, \\ 2(t - \frac{3}{2})^2 + \frac{1}{2}, & 1 \le t \le 2. \end{cases}$$

Hence $1/2 \le |\gamma(t)|^2 \le 1$, with $\gamma(0) = 1$ and $\gamma(2) = -1$. Choose $c > 0$ with $c^2 < r$. The path $w(t) = c^2\gamma(t)^2$ lies in $D_r \setminus \{0\}$ and has $w(0) = w(2) = c^2$. If S existed, the continuous function

$$q(t) = \frac{S(w(t))}{c\gamma(t)}$$

would satisfy $q(t)^2 = 1$. Connectedness of $[0, 2]$ makes q constant. But its numerator agrees at the endpoints and its nonzero denominator changes sign, so $q(2) = -q(0)$, a contradiction. □

Proof of Theorem 1. Restrict coefficients by the continuous embedding

$$\iota(u + iv) = (0, 0, -u, 0, 0, -v/2).$$

Equations (1)–(2) then become $x^2 - y^2 = u$ and $2xy = v$, so their full real solution set is exactly the complex roots $z = x + iy$ of $z^2 = u + iv$. If U were an admissible neighborhood of zero, $\iota^{-1}(U)$ would be an open neighborhood of zero in $\mathbb{C}$ and would contain some D_r. Composing its exact selection with ι, and identifying $\mathbb{R}^2$ with $\mathbb{C}$, would contradict Lemma 1. Every open cover has a member containing zero, so its cardinality cannot evade this obstruction. □

Remark 1 (Translated obstructions)**.** *For any $h, k \in \mathbb{R}$, define*

$$p_{h,k} = (-2h,\ 2k,\ h^2 - k^2,\ -k,\ -h,\ hk).$$

No open neighborhood of $p_{h,k}$ is admissible. Indeed, replacing the constant coefficients by $a_0 = h^2 - k^2 - u$ and $b_0 = hk - v/2$, while fixing the linear coefficients, gives

$$(x - h)^2 - (y - k)^2 = u, \qquad 2(x - h)(y - k) = v.$$

The same argument applies to $z = (x - h) + i(y - k)$. In particular $p_{1,0} = (-2, 0, 1, 0, -1, 0) \neq 0$ is obstructed. Removing just the zero coefficient vector therefore does not yield an admissible open cover of the remaining coefficient space.

3 Provenance and the limits of the answer

The final journal statement [1] and its arXiv version 1504.01997v1 both ask for all of $\mathbb{R}^6$, ordinary open domains, and continuous exact roots. The source itself identifies the complex-square slice. Its finite lower-bound hint does not remove zero. For $z^2 = w$ over $\mathbb{C} \setminus \{0\}$, two slit domains support branches and one global branch is impossible. Including zero prevents even a local exact branch. This is the domain distinction responsible for Theorem 1; no intention to alter the source is attributed to its author.

The strict-cover framework and the obstruction mechanism are classical. Smale's definition uses open sets with strict right inverses, while his root-selection Theorem A removes the discriminant [2, pp. 83–85]. Pavešić explicitly distinguishes ordinary open strict sections from filtered domains and homotopy sections, and warns that failure of local sectionability defeats open section covers [3, Section 5, pp. 75–76][4, Section 2, pp. 109,111]. We credit that machinery and Vassiliev's slice; no new topological method is claimed.

A bounded primary-body priority audit did not locate an earlier explicit diagnosis of this exact six-coefficient formulation. It cannot establish novelty or historical firstness. Important coverage limits include unread Schwarz originals/translation and Vassiliev's 1996 article; the complete 2011 author manuscripts were read without comparison to their final journal versions. Other version and selective-read limits are recorded in the accompanying priority audit. This note records the explicit literal answer, without a priority guarantee.

Deleting an appropriate coefficient locus, changing admissible domains, or using homotopy sections would pose different questions. Theorem 1 does not solve any unspecified repaired finite-genus problem. It also does not solve approximate Problem 2B in [1]: on the square slice, for fixed $\varepsilon > 0$, the value zero is within ε of every root whenever $|w| < \varepsilon^2$.

Verification and disclosure

The supplementary package provides exact rational polynomial checks of the slice identities, the full Jacobian, the path's universal quadratic norm bounds, and adversarial input controls. The checks remain active with Python optimization. They do not prove continuity or the topological nonexistence statement; those are established by the written argument above. Finite numerical sampling is not used as a proof of the path bound.

AI tools were used extensively in developing, drafting, sourcing, reproducing, and adversarially auditing this work. This preprint is unrefereed and has not undergone conventional human peer review. The author is responsible for the claims and their stated limits.

References

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[4] P. Pavešić, *Topological Complexity of a Map*, Homology Homotopy Appl. **21**(2) (2019), 107–130. doi:10.4310/HHA.2019.v21.n2.a7. The relevant final-body passages were checked separately from the later corrected arXiv:1809.09021v3.