# Negative answers to the printed totally mixed face questions

Alec Kriebel*

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**Abstract**

For two polytopes, retain the faces of their Minkowski sum whose two canonical summands have positive dimensions and whose dimension is the sum of those dimensions. This is the definition in Payne's printed Oberwolfach question. Let $D$ be the standard four-simplex and $I = [0, 1]$. For the full-dimensional lattice polytopes $D \times I$ and $(-D) \times I$ in $\mathbb{R}^5$, the qualifying-face order complex is a disjoint union of two connected pure graphs, each with 50 vertices and 60 edges. Its rational Betti numbers are $(2, 22, 0, \ldots)$. Thus both printed assertions, shellability and the homology of a wedge of three-spheres, fail. The shared interval factor deliberately produces a nongeneric example. The argument concerns the printed additive-dimension definition; it does not settle shellability for the different mixed faces used in stable tropical intersections.

## 1 The exact question and the example

Payne's question [1, p. 2464, Problem 3] concerns polytopes $P_1, P_2 \subset \mathbb{R}^n$ and those faces $F = F_1 + F_2$ satisfying

$$\dim F_i > 0 \quad (i = 1, 2), \qquad \dim F = \dim F_1 + \dim F_2. \tag{1}$$

It asks whether their inclusion poset is shellable and whether its realization has the rational homology of a wedge of $(n-2)$-spheres. There is no stated general-position assumption. We use the standard order complex: vertices are qualifying faces and simplices are nonempty strict chains. No artificial extrema are added.

For a face exposed by a functional $u$, its canonical summands are the faces $P_i^u$ exposed by the same $u$. The standard uniqueness theorem for Minkowski-sum faces appears, for example, in [2, Lemma 4 of the cited arXiv version]. It also removes an existential ambiguity in (1): if nonempty summand faces $G_i$ have $G_1 + G_2 = P_1^u + P_2^u$, exposure first forces $G_i \subseteq P_i^u$. For every other functional, the two nonnegative deficits between their support functions sum to zero. Each deficit is therefore zero, so $G_i = P_i^u$.

**Theorem 1.** *Let $D = \operatorname{conv}(0, e_1, e_2, e_3, e_4) \subset \mathbb{R}^4$ and $I = [0, 1]$. For $P_1 = D \times I$ and $P_2 = (-D) \times I$, let $T$ be the inclusion poset selected by (1). Its order complex is a disjoint union $K \sqcup K$, where $K$ is a connected pure graph with 50 vertices and 60 edges. In particular,*

$$H_0(|T|; \mathbb{Q}) = \mathbb{Q}^2, \qquad H_1(|T|; \mathbb{Q}) = \mathbb{Q}^{22}, \qquad H_j(|T|; \mathbb{Q}) = 0 \quad (j \geq 2).$$

*The poset is not shellable, under either pure or standard nonpure shellability, and its realization does not have the rational homology asked for in the printed question.*

*Independent researcher. ORCID: 0009-0001-9320-500X.

Both summands are full-dimensional lattice polytopes in $\mathbb{R}^5$. The proof uses neither empty realizations nor low-dimensional sphere conventions. The simplex-difference geometry is classical [2, 3]; the observation used here is the effect of a common interval factor on the exact filter (1).

## 2 Complete face calculation

Write $v_0 = 0$, $v_i = e_i$, and $E = \{0, 1, 2, 3, 4\}$. Every nonempty proper face of $D - D$ is uniquely of the form

$$F(A, B) = \operatorname{conv}\{v_i : i \in A\} + \operatorname{conv}\{-v_j : j \in B\}, \qquad \varnothing \neq A, B \subset E, \quad A \cap B = \varnothing.$$

Indeed, for a nonzero functional $u$, take $A$ and $B$ to be its maximum and minimum index sets. They are disjoint because the vertices affinely span $\mathbb{R}^4$. Conversely, assign values $1, -1, 0$ on $A, B$ and the remainder, and subtract the value at $v_0$ to obtain a linear functional with exactly these extrema.

The two summand direction spaces intersect only at zero: a vector in their intersection gives an affine dependence on the disjoint vertex sets, with coefficient sum zero on each set. Affine independence makes every coefficient zero. Consequently,

$$\dim F(A, B) = |A| + |B| - 2, \qquad \dim D^u = |A| - 1, \quad \dim(-D)^u = |B| - 1. \tag{2}$$

Moreover,

$$F(A, B) \subseteq F(C, L) \quad \Longleftrightarrow \quad A \subseteq C,\ B \subseteq L. \tag{3}$$

For the nontrivial direction, evaluate every $v_i - v_j$, $i \in A, j \in B$, against a functional exposing $F(C, L)$. Attaining the maximum difference forces $i \in C, j \in L$. The reverse direction follows directly from convex hulls.

Thus the base qualifiers have sizes $(|A|, |B|) = (2, 2), (2, 3), (3, 2)$. There are $\binom{5}{2}\binom{3}{2} = 30$ minimal elements of type $(2, 2)$ and $2\binom{5}{2} = 20$ maximal elements. Each maximum has three lower minima, and each minimum has two upper maxima, obtained by placing its one unused index on either side. There are no three-element strict chains. The base order complex $K$ is therefore a pure graph on 50 vertices with 60 edges.

To check connectedness, suppress the degree-two minimal vertices. The resulting graph has signed two-subsets $A^+, A^-$ of $E$ as vertices, with $A^+$ adjacent to $B^-$ exactly when $A \cap B$ is empty. Two distinct intersecting same-sign subsets have the opposite-sign complement of their union as a common neighbor. If two same-sign subsets are disjoint, a two-subset containing one element of each intersects both; apply the preceding path twice. All vertices of a fixed sign therefore connect, and every vertex has a neighbor of the other sign. The graph, and hence its subdivision $K$, is connected. Contracting a spanning tree shows that $K$ is homotopy equivalent to a wedge of $60 - 50 + 1 = 11$ circles.

Now $P_1 + P_2 = (D - D) \times [0, 2]$. A functional $(u, t)$ with $t > 0$ or $t < 0$ exposes an upper or lower endpoint face; its summand dimensions and qualification are exactly the base ones. If $t = 0$, write $a, b$ for the base summand dimensions and $c$ for the base sum dimension. The two summands then contain the interval factor, so their dimensions sum to $a + b + 2$, while their sum has dimension $c + 1 \leq a + b + 1$. No such face passes (1). If $u = 0, t \neq 0$, the additional horizontal end face has dimension 4 but both summands have dimension 4, so it also fails. The whole sum has dimensions $(5, 5, 5)$ and fails; the empty face fails positivity.

These cases exhaust all faces. The qualifying poset is precisely two endpoint copies of the base poset, with no cross-comparisons. This proves the asserted decomposition and homology.

In any proposed shelling, the first edge from the component different from the initial edge has empty intersection with the preceding facets. That intersection has dimension $-1$, whereas a new edge requires a pure zero-dimensional intersection [4, Definition 2.1]. This rules out both shellability conventions. Finally, a wedge of three-spheres, including the zero-fold point, is connected and has no first homology. The same obstruction would persist if one replaced the printed sphere dimension by $n-3$.

## 3 Related results, scope, and verification

For comparison, $D_3$ and $-D_3$ in $\mathbb{R}^3$ yield just six incomparable qualifying faces: the index pairs partition four vertices into two pairs. Their order complex is six points. This refutes the printed wedge-of-circles homology assertion but is shellable in dimension zero. Scaled opposite tetrahedra and their six-ray geometry already occur in Sturmfels–Tevelev [3, Examples 1.3 and 4.10]; this lower-dimensional control is not presented as new. The five-dimensional example is needed here for nonshellability.

The term mixed has several different definitions. In the published Adiprasito et al. paper [5, Section 1.2], totally mixed faces require facets of the summands. That counting theorem is not used. The public 2020 manuscript underlying Adiprasito's 2024 article [6, p. 2, Corollary 1.3 of the public manuscript] instead selects faces satisfying

$$\dim\Big(\sum_{i\in J} F_i\Big) \geq |J| \quad \text{for every nonempty index subset } J.$$

For two summands this is broader than (1). For instance, a proper vertical face $F(A,B)\times[0,2]$ with $|A|=|B|=2$ has summand dimensions $2,2$ and sum dimension 3. It meets that lower-bound criterion but fails additivity. Such vertical faces are absent from our poset. The counterexample therefore does not contradict the different stable-intersection theorem, or resolve its shellability question. Related tropical link obstructions occur in Hiatt–Simpson–Wang–Wu's revised 2026 construction [7, Section 8]. Its Schön variety in $(\mathbb{C}^*)^8$ is a different object; no identification with this two-summand exact face poset is supplied or assumed here.

The supplement records a bounded primary-source priority audit, including current versions and related tropical results. No earlier counterexample or theorem settling nonshellability for this exact poset was identified. Such a result, together with the classical homology antecedent, would already settle both assertions. This is not an exhaustive literature or first-ever priority certificate. The 2024 journal metadata for [6] was verified, but its paywalled final body was not inspected; the definition comparison above is explicitly with the public 2020 manuscript.

Two independently authored standard-library programs accompany the analytic proof. One reconstructs facets and their intersection lattice from integer coordinates, all 543 nonempty sum faces, and all 8649 summand-face pairs. The other independently constructs the order complex and its signed rational boundaries, obtaining boundary rank 98. Both run with explicit exceptions in ordinary and optimized Python; positive and adverse controls are included. These computations are supplementary, and the proof above requires no program.

**AI and review disclosure.** AI tools were used extensively in developing, verifying, researching, and writing this result and its supporting materials. This preprint is unrefereed. Independent adversarial AI reviews do not constitute external human peer review, journal acceptance, or formal proof-assistant certification.

## References


[1] S. Payne, *The complex of totally mixed faces*, Problem 3 in J. Doolittle (collector), Open Problems, Oberwolfach Report 39/2019, p. 2464. doi:10.4171/OWR/2019/39.

[2] K. Fukuda and C. Weibel, *On $f$-vectors of Minkowski additions of convex polytopes*, Discrete Comput. Geom. **37** (2007), 503–516. arXiv:math/0510470v4.

[3] B. Sturmfels and J. Tevelev, *Elimination theory for tropical varieties*, Math. Res. Lett. **15** (2008), 543–562. arXiv:0704.3471.

[4] A. Björner and M. L. Wachs, *Shellable nonpure complexes and posets. I*, Trans. Amer. Math. Soc. **348** (1996), 1299–1327. doi:10.1090/S0002-9947-96-01534-6.

[5] K. Adiprasito, P. Brinkmann, A. Padrol, P. Paták, Z. Patáková, and R. Sanyal, *Colorful simplicial depth, Minkowski sums, and generalized Gale transforms*, Int. Math. Res. Not. **2019**(6), 1894–1919 (online 2017). doi:10.1093/imrn/rnx184.

[6] K. Adiprasito, *A note on the topology of Minkowski sums and complete intersections*, Pure Appl. Math. Q. **20**(1) (2024), 17–28. doi:10.4310/PAMQ.2024.v20.n1.a2. Inspected public manuscript: *A note on the topology of polyhedral hypersurfaces and complete intersections*, July 27, 2020, author PDF.

[7] S. Hiatt, C. Simpson, B. Wang, and C. Wu, *Topology and Euler characteristics of tropical varieties*, arXiv:2606.10817v2, July 24, 2026. arXiv:2606.10817v2.