

NP-completeness of three-terminal shortest-path blocking

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Abstract

We prove that, in a finite simple undirected graph with unit edge lengths, finding a minimum edge set whose deletion increases all three distances between three specified terminals by at least one is NP-hard. The budget decision problem is NP-complete even when all deletion costs are one and all three original terminal distances are three. A direct reduction from Vertex Cover passes through a tripartite graph and identifies its vertex-cover number with the minimum shortest-path blocker size. The construction also gives NP-completeness when the terminal distances must remain finite, or must increase exactly from three to four. This provides a classification of the undirected triangle question posed by Krause in 2005, without asserting bibliographic first priority.

1 The question and its model

Krause asked for the complexity of Blocking Shortest Paths (BSP) when the demand pairs are the three pairs among terminals r, s, t [1, pp. 2856–2858]. The same undirected triangle case appears in his dissertation [2, Section 4.3]; directed triangle hardness was already known [2, Theorem 9.3]. We classify the undirected problem by a self-contained reduction, using only the classical NP-completeness of Vertex Cover [3] as an external complexity result.

An earlier related construction attaches the three parts of a tripartite Vertex Cover instance to three vertices on a terminal path [4, Section 6]. That construction concerns protected vertices, vertex deletion, one terminal pair and zero/one traversal lengths. We retain the tripartite attachment mechanism but use three geodesic demands, unit traversal lengths, and unrestricted unit-cost edge deletion. The no-larger-cover extraction below explicitly accounts for deletion of the middle edges.

All graphs below are finite, simple and undirected, and every edge has traversal length one. Write $d_G(x, y)$ for shortest-path length, with $d_G(x, y) = \infty$ if there is no path. The three terminals are distinct and pairwise connected initially. A *blocker* is an edge set F satisfying

$$d_{G-F}(x, y) \geq d_G(x, y) + 1 \quad \text{for } \{x, y\} \in \{\{r, s\}, \{r, t\}, \{s, t\}\}. \quad (1)$$

Terminal disconnection is allowed: infinity exceeds every finite integer. Deleting edges cannot shorten a path, so (1) is equivalent to meeting every original shortest path for each demand pair. All edges, including terminal edges, may be deleted. Deletion costs and traversal lengths are different input attributes.

The unit-cost decision problem supplies a nonnegative, binary-encoded integer budget B and asks whether $|F| \leq B$. An edge-list certificate and three breadth-first searches before and after deletion verify feasibility in polynomial time. With positive binary-encoded rational deletion costs and a rational budget, exact addition and comparison have polynomial bit complexity as well. Thus both decision versions belong to NP.

2 Two cover identities

For a graph J , let $\tau(J)$ be its minimum vertex-cover cardinality. The first identity is the vertex-cover form of Poljak's classical two-subdivision stable-set identity [5, p. 308]; we include its proof.

Lemma 1. *Let $Q = (V, E)$ have n vertices and m edges. Replace each edge e with ordered endpoints (u_e, v_e) by the path*

$$u_e - b_e - c_e - v_e,$$

using two new vertices unique to that edge. The resulting graph H is tripartite with parts $A = V$, $B = \{b_e : e \in E\}$ and $C = \{c_e : e \in E\}$, and

$$\tau(H) = m + \tau(Q).$$

If $m \geq 1$, there is an edge between each pair of parts.

Proof. Given a cover S of Q , add one new vertex per edge: choose c_e if only u_e is in S , choose b_e if only v_e is in S , and choose either if both are. The resulting set covers H and has size $m + |S|$.

Conversely, let X cover H , and put $S = X \cap V$. Every middle edge requires at least one of b_e, c_e in X . If neither original endpoint is in S , both new vertices are required. Writing q for the number of such uncovered Q -edges gives $|X| \geq |S| + m + q$. Adding one endpoint of each uncovered edge to S produces a cover S' of Q with

$$\tau(Q) \leq |S'| \leq |S| + q \leq |X| - m.$$

These inequalities prove the identity. Each edge gadget contributes one edge of each cross-part type, establishing the final assertion. $\square$

Lemma 2. *Let $H = (W, J)$ be tripartite with a supplied partition $W = A \sqcup B \sqcup C$, and with at least one edge of each cross-part type. Add new terminals r, s, t , adjacent respectively to all vertices in A, B, C , and no other new edges. Call the resulting graph G_H . Its three terminal distances are three, and its minimum unit-cost blocker size satisfies*

$$\text{opt}(G_H) = \tau(H).$$

Proof. For $w \in W$, let e_w be its unique terminal spoke. Terminals are nonadjacent and have disjoint neighborhoods, ruling out terminal paths of length one or two. A cross-part edge supplies a path of length three for each terminal pair. Moreover, for the terminals a, b of parts U, V , the complete family of shortest paths is

$$a - u - v - b \quad (u \in U, v \in V, uv \in J).$$

Indeed, any three-edge path must use a spoke at each end and one H -edge between its two interior vertices.

If X covers H , deleting the spokes e_w for $w \in X$ meets every such geodesic. Therefore $\text{opt}(G_H) \leq \tau(H)$.

For the reverse inequality, let F be any blocker, allowing deletions of H -edges as well as spokes. Include w in a set X for every deleted spoke e_w , and include one arbitrary endpoint of every deleted H -edge. Then $|X| \leq |F|$. For each $uv \in J$, the corresponding terminal geodesic has edges e_u, uv, e_v . At least one is in F , and its contribution to X covers uv . Thus X is a vertex cover and $\tau(H) \leq |F|$. $\square$

3 The complexity classification

Theorem 3. *Undirected, unit-length three-terminal BSP is NP-complete in its budget decision form, even with unit deletion costs and all three original terminal distances equal to three. Exact optimization is NP-hard. The positive binary-encoded rational-cost decision version is also NP-complete.*

Proof. Take a Vertex Cover instance (Q, k) with $k \geq 0$ and $m = |E(Q)| \geq 1$. Construct H and then G_H by the two lemmas, and set $B = m + k$. They give

$$\tau(Q) \leq k \iff \tau(H) \leq m + k \iff \text{opt}(G_H) \leq B.$$

The graph has $n + 2m + 3$ vertices and $n + 5m$ edges. Construction is polynomial in the input length, and computing $m + k$ takes polynomial bit time. The graph size does not depend on the magnitude of the binary budget. The hypotheses of Lemma 2 hold because $m \geq 1$. In particular the terminals are pairwise nonadjacent, initially connected, and at distance three.

Edgeless source instances are always yes-instances and can be sent to a fixed yes-instance satisfying the restrictions: use a triangle as H , attach one spoke per vertex, and set $B = 2$. Thus the reduction is total. Vertex Cover NP-completeness and the NP-membership argument above prove the unit-cost claim. Unit costs are permitted positive rational costs, giving hardness of that version too. An exact optimizer decides the budget problem by comparing its optimum with the budget, proving optimization hardness. $\square$

Corollary 4. *The unit-cost decision problem remains NP-complete if all terminal distances must remain finite, or must increase exactly from three to four.*

Proof. Add to G_H a fresh internally vertex-disjoint four-edge path between each terminal pair; the nine new internal vertices have no other incident edges. A simple terminal path using a new internal vertex must traverse a whole added path. Thus no new terminal path of length at most three is created, and the original terminal geodesics are unchanged.

Any blocker in the augmented graph must still hit each original geodesic. Ignoring backup-path deletions, the extraction in Lemma 2 produces an H -cover of no greater size. Conversely, spoke deletions for an H -cover leave all three backup paths intact and block every length-three geodesic, so all terminal distances become exactly four. Both strengthened optima are therefore $\tau(H)$, and the same budget reduction applies. Their additional distance constraints are verifiable by BFS, proving NP membership. This asserts terminal connectivity, not connectivity of every remaining vertex. $\square$

4 Verification, priority and research status

The accompanying source and verification archive preserves the original proof packet and its hash manifests, along with independent mathematical, source-model, computational and priority audits.

The standard-library Python verification checks include arbitrary edge-deletion subsets, cover extraction, tripartite optimum equality, all nonempty labelled source graphs through five vertices, binary budgets, and backup-path controls. Integrity mutations are rejected. Finite checks support consistency and reproducibility; the universal theorem follows from the written lemmas, not enumeration or a proof assistant.

The result answers the mathematical question recorded as open in the historical sources. A bounded current-literature audit is supplied separately. We make no claim that the problem remained open until this work, that our argument is the first proof, or that the literature search is exhaustive. General length-bounded cuts, directed demands and full-disconnection multicut require separate model comparisons; their hardness alone does not establish this restricted theorem.

AI tools were used extensively in constructing, verifying, adversarially auditing and preparing this work. Independent AI-agent review is disclosed as such. This is an unrefereed preprint: it has not undergone independent human peer review or journal review. No planar, bounded-degree, weighted-traversal-length or globally connected residual-graph classification is claimed.

References

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