# An explicit counterexample to three-summand Kato-kernel splitting

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**Abstract**

Question 2 in Laghribi's contribution to Oberwolfach Report 5/2016 asks whether a mixed Kato kernel is the sum of the kernels of a finite purely inseparable extension, a separable quadratic extension, and a bilinear Pfister function field. We give an explicit negative instance in $H^2_2$, using an exponent-two extension and a binary bilinear Pfister form that remains anisotropic. The mixed class is exactly a specialization of the generators of Aravire–Laghribi–O'Ryan (2019/2020). We prove directly that it remains outside the sum after the additional Pfister kernel is included, by a universal norm and residue obstruction. The original degree-one cohomology case is separately positive. No firstness claim is made.

## 1 The question and the counterexample

For a field $E$ of characteristic two, put

$$H^{n+1}_2(E) = \Omega^n_E/(\wp(\Omega^n_E) + d\Omega^{n-1}_E), \qquad H^1_2(E) = E/\wp(E).$$

Here $\wp$ on logarithmic generators is induced by $u\,d\log v_1 \wedge \cdots \mapsto (u^2+u)\,d\log v_1 \wedge \cdots$, modulo exact forms. Write $\mathcal{K}_n(E/F)$ for the kernel of $H^{n+1}_2(F) \to H^{n+1}_2(E)$, and abbreviate $\mathcal{K}_1(E/F) = \mathcal{K}(E/F)$.

The original question [1, printed p. 251, Question 2] asks whether

$$\mathcal{K}_n(LK(B)/F) = \mathcal{K}_n(L/F) + \mathcal{K}_n(K/F) + \mathcal{K}_n(F(B)/F) \tag{1}$$

for every finite purely inseparable $L/F$, separable quadratic $K/F$, and anisotropic bilinear Pfister form $B$ remaining anisotropic over $L$. As in that source, $F(B)$ is the function field of the *projective quasilinear quadric* of $v \mapsto B(v,v)$. This convention includes an anisotropic binary form, whose quadric is integral and zero-dimensional.

The positive exponent-one regime is covered by [2]. Higher-exponent two-field nonsplitting and the mixed generators are antecedent work [3]. The class below is exactly their Definition 1.3 and Lemma 3.1 specialization: in their notation take $n = m = 1$, $f_0 = 1$, $f_1 = z$, and $a = s^{-4}$. The only nonzero generator term is $(\beta^2 z/(1+\beta^2 z))^2 d\log z$. Moreover $s^{-4} + s^{-1} = \wp(s^{-2} + s^{-1})$, so these parameters define the same separable quadratic field. This note supplies the direct three-summand nonmembership proof; it makes no priority claim for the class, example, or target deduction.

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**Theorem 1.** *Let $b, z, s, t$ be algebraically independent over $\mathbb{F}_2$, and set*

$$F = \mathbb{F}_2(b, z, s, t), \quad L = F(\beta), \quad \beta^4 = b, \qquad K = F(\theta), \quad \theta^2 + \theta = s^{-1}, \qquad B = \langle 1, t \rangle_b.$$

*Then $L/F$ has degree four and exponent two, $K/F$ is separable quadratic, and $B_L$ is anisotropic. The class*

$$\xi = \overline{s^{-4} \frac{bz^2}{1 + bz^2} \, d\log z} \ \in H^2_2(F)$$

*satisfies*

$$\xi \in \mathcal{K}(LK/F) \subseteq \mathcal{K}(LK(B)/F), \qquad \xi \notin \mathcal{K}(L/F) + \mathcal{K}(K/F) + \mathcal{K}(F(B)/F).$$

*Consequently* (1) *is false at its allowed index $n = 1$.*

We use the natural identification $H^2_2(E) \simeq \mathrm{Br}(E)[2]$, sending $u \, d\log v$ to the characteristic-two quaternion class $[u, v)_E$; see [4, pp. 2–3]. Its generators satisfy $i^2 + i = u$, $j^2 = v$, $ji = (i+1)j$. For a genuine Artin–Schreier quadratic field $E(\eta)/E$, $\eta^2 + \eta = u$, the cyclic relative Brauer theorem says

$$\mathcal{K}(E(\eta)/E) = \{[u, c)_E : c \in E^*\}.$$

Indeed a normalized two-cocycle of the order-two Galois group is specified by an invariant scalar $c$; coboundaries multiply it by a norm. Its crossed product is the displayed quaternion algebra [5, IV, §3, items 3.11–3.14]. Also $[u, c)_E = 0$ if and only if $c$ is a norm from $E(\eta)^*$ [6, Theorem 6.4.11]. These statements allow imperfect base fields.

## 2 Proof of Theorem 1

The $b$-valuation shows that $b$ is nonsquare, so $X^4 - b$ is irreducible and $L = \mathbb{F}_2(\beta, z, s, t)$ has the claimed degree and exponent. A negative valuation of $x^2 + x$ is even; hence the odd $s$-pole of $s^{-1}$ proves the Artin–Schreier extension is genuine. The $t$-valuation in $L$ proves $t \notin L^2$, giving anisotropy. It remains nonsquare in separable algebraic extensions of $L$.

Let $\gamma^2 = t$. The quadric $X^2 + tY^2 = 0$ in $\mathbb{P}^1$ is entirely in the $Y \neq 0$ chart, with field coordinate ring $F[X/Y]/((X/Y)^2 + t)$. Thus

$$F(B) = F(\gamma), \qquad M := LF(B) = \mathbb{F}_2(\beta, z, s, \gamma), \qquad LK(B) = M(\theta).$$

The same descriptions over $K$ are valid since a separable extension cannot contain a new purely inseparable square root.

**Vanishing in the composite.** In $L$, put $f = 1 + \beta^2 z$ and $h = \beta^2 z / f$. Since $d(\beta^2) = 0$,

$$d\log f = h \, d\log z, \qquad h^2 = \frac{bz^2}{1 + bz^2}.$$

Apply the Artin–Schreier relation first to $s^{-2} h \, d\log z$, and then to $s^{-1} d\log f$. In cohomology,

$$\xi_L = \overline{s^{-2} h \, d\log z} = [s^{-2}, f)_L = [s^{-1}, f)_L. \tag{2}$$

This is zero over $LK$. The equalities occur in the quotient by exact forms and Artin–Schreier images, not by treating $\wp$ as $L$-linear.

**Reduction of all three summands.** Suppose $\xi = x + y + w$, with the three individual kernel memberships in Theorem 1. The cyclic description gives $y = [s^{-1}, c)_F$ for some $c \in F^*$. Restrict to $M = LF(B)$: both $x$ and $w$ vanish. By (2) and multiplicativity of the second symbol slot,

$$[s^{-1}, f/c)_M = 0. \tag{3}$$

Thus the extra Pfister summand is fully included in the argument.

**A necessary residue condition for every $c$.** Put $k = \mathbb{F}_2(b, z, t)$, $k' = \mathbb{F}_2(\beta, z, \gamma)$. Complete $M = k'(s)$ to $D = k'((s))$. The odd-pole argument also proves that $T = D(\theta)$ is a quadratic field. With $\pi = \theta^{-1}$,

$$\pi^2 + s\pi + s = 0, \qquad \mathrm{N}_{T/D}(\pi) = s, \qquad \mathrm{N}_{T/D}(A + B\pi) = A^2 + sAB + sB^2.$$

Here the conjugate of $\pi$ is $\pi + s$. For nonzero $A, B \in D$, let $a = v_s(A)$, $b' = v_s(B)$. The even integer $2a$ and the odd integer $1 + 2b'$ differ. The middle-term valuation $1 + a + b'$ is strictly greater than their minimum: split into $a \le b'$ and $a \ge b' + 1$. Hence the norm has valuation $\min(2a, 1 + 2b')$. The cases $A = 0$ or $B = 0$ follow from the same formula. It follows that a unit norm has $v_s(A) = 0$, $v_s(B) \ge 0$, and its residue is a square in $k'$. This proof uses no perfectness or defect assumption.

For arbitrary $c \in F^* = k(s)^*$, write $c = s^r c_0(1 + O(s))$ with $r \in \mathbb{Z}$, $c_0 \in k^*$. Equation (3) makes $f/c$ a norm over $D$. Multiplication by the norm $s^r = \mathrm{N}(\pi^r)$, valid also for negative $r$, makes it a unit norm. Its residue therefore satisfies

$$\frac{1 + \beta^2 z}{c_0} \in k'^{*2}. \tag{4}$$

**The residue condition is impossible.** Set

$$S = k'^2 = \mathbb{F}_2(\beta^2, z^2, \gamma^2), \qquad R = \mathbb{F}_2(\beta^4, z^2, \gamma^2).$$

We have $k = R(z)$, of degree two over $R$, so $c_0 = u + vz$ uniquely with $u, v \in R$, including arbitrary rational denominators. Also $z \notin S$, as seen by the odd valuation of $z^2$ in the rational field $S$. Comparing $1, z$ coefficients in

$$1 + \beta^2 z = q(u + vz), \qquad q \in S^*,$$

gives $qu = 1$, $qv = \beta^2$. Thus $u \neq 0$ and $\beta^2 = v/u \in R$, a contradiction: as an independent variable, $\beta^2$ has valuation one at zero, whereas nonzero elements of $R = \mathbb{F}_2((\beta^2)^2, z^2, \gamma^2)$ have even valuation there. This disproves (4) for every $c$, proving the theorem. □

## 3 The boundary degree and scope

**Proposition 2.** *At $n = 0$, equality* (1) *holds for every set of its stated data with $B$ a positive-fold bilinear Pfister form.*

*Proof.* An Artin–Schreier root cannot first appear in a purely inseparable extension, since that root is separable over the base. Nor can it first appear in a rational function field, whose algebraic constants are its base field. An anisotropic quasilinear form of dimension $d \ge 2$ has projective-quadric function field purely inseparable quadratic over a rational field in $d - 2$ variables: use a projective chart and solve for one squared coordinate. For $d = 2$, there are no rational variables.

Separable extension preserves quasilinear anisotropy, by linear disjointness with purely inseparable extensions. Thus the chart description applies over $KL$. Any Artin–Schreier root over $F$ in $LK(B)$ descends first to $KL$, then to $K$, since $KL/K$ is purely inseparable. Hence the composite kernel is the $K/F$ kernel, and the other two kernels are zero. □

If one separately assigns $F(B) = F$ to a zero-fold form, its $n = 0$ case follows from the same purely inseparable injection. No function field of an empty projective quadric is needed.

Two boundaries of the construction are useful. If the Pfister slot is $z$ rather than the independent $t$, adjoining its field kills $dz$ and absorbs $\xi$. If $b = u^2$ already in the base and $\beta^2 = u$, then the computation *over* $L$ gives $\xi_L = [s^{-1}, 1 + uz)_L$. Since $1 + uz$ is now in the base, this proves membership in $\mathcal{K}(L/F) + \mathcal{K}(K/F)$. It does not justify a single-symbol equality in the base: there $d(1 + uz) = u\,dz + z\,du$. This corrects an auxiliary shortcut in the earlier draft and leaves the exponent-two example unchanged.

**Verification, credit and review status.** The supplement gives exact algebra diagnostics, including the fixed-logarithmic-basis exact-differential correction, and records the mathematical and bounded primary priority audits. Finite checks supplement the universal proof; they are not a cohomology decision procedure. The audits did not verify an earlier recorded full three-summand answer in the bodies inspected. Final journal-body comparison for [3] and the full bodies of the adjacent nonnormal-extension paper (DOI 10.1080/00927872.2019.1686510) and 2022 transfer paper (DOI 10.1016/j.jpaa.2021.106930) remain access limits. No global openness, firstness, or causal-independence certificate is asserted. AI tools were used extensively in developing, writing, reproducing and adversarially checking this work. Initial internal review families had parent context available; their checks were not perfectly isolated. This preprint is unrefereed, has not received external human peer review, and is not formally proof-assistant verified.

## References


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