A three-point counterexample to a random lattice discrepancy comparison

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Abstract

For three points $\{\Delta\}, \{Z+\Delta\}, \{2Z+\Delta\}$ with independent continuous uniform generator Z and shift Δ, the expected normalized anchored fourth discrepancy moment is $19/1620$. The corresponding independent-point moment is $4/405 = 16/1620$. This gives a negative answer to the universal expected-moment comparison posed by Novak, reporting joint work with Hinrichs, in 2004. We give an elementary inclusion-area proof and a separate empty-tail argument giving reversal for every point count $n \geq 4$ in dimension one. Three is the smallest possible counterexample point count. Ferudun's preprint of 14 October 2026 also gives the same exact counterexample and broader results; this note records the proof in the source repository PR dated 10 October. First priority is not claimed.

1 The comparison and its recorded scope

For indexed points $t_0, \ldots, t_{n-1} \in [0,1)^d$, define

$$\operatorname{disc}_p(t)^p = \int_{[0,1]^d} \left| \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_{[0,x)}(t_j) - \prod_{r=1}^{d} x_r \right|^p dx.$$

Novak, reporting joint work with Hinrichs, asks whether, for even p, the expectation of this quantity for

$$M_n^{Z,\Delta} = (\{jZ+\Delta\})_{j=0}^{n-1}, \qquad Z, \Delta \text{ independent uniform on } [0,1)^d,$$

is at most the expectation for n independent uniform points [1, p. 698]. Fractional parts are coordinatewise. This is the full continuous generator-and-shift average, with no selection of good generators. A single reversed triple (n, d, p) disproves the universal comparison.

Theorem 1. *In dimension one,*

$$\mathbb{E}\operatorname{disc}_4(M_3^{Z,\Delta})^4 = \frac{19}{1620} > \frac{4}{405} = \mathbb{E}\operatorname{disc}_4(M_3^{\mathrm{iid}})^4.$$

The excess is $1/540$ *and the ratio is* $19/16$. *For* $n \leq 2$ *the complete lattice and independent-point ensemble laws agree in every dimension. Thus three is the smallest possible point count for a counterexample.*

Related result and chronology. Ferudun's unrefereed preprint [2, Theorems 1.2–1.3], released on 14 October 2026, contains the same three-point values, the same empty-progression mechanism, and broader results in dimension and exponent. It is an explicit overlapping answer and is credited here. The source proof of the present note is in repository PR 1135, whose GitHub creation record is dated 10 October 2026 [3]; the present note is a new release on 17 October. These records do not certify historical public visibility of the repository, first discovery, or first publication. No inference about either author's awareness or dependence is made. The bounded literature audit is supplied with the verification package; it identified no earlier full answer but does not certify exhaustive priority. The elementary proof below is self-contained.

2 The three-point calculation

Set $U = t_0 = \Delta$ and $V = t_1 = \{Z + \Delta\}$. These are independent uniform variables, since translation preserves uniform circle measure, and $t_2 = \{2V - U\}$. More generally, every two distinct indexed points are independent uniforms: conditional on Z, t_j is uniform, so t_j is independent of Z; multiplication by the nonzero integer $k - j$ preserves circle measure, and $t_k = \{t_j + (k-j)Z\}$. This argument also applies coordinatewise.

For $0 \le x \le 1/2$, put

$$q(x) = \mathbb{P}(t_0 < x,\ t_1 < x,\ t_2 < x).$$

On $U, V \in [0, x)$, one has $-x < 2V - U < 2x$. A negative value wraps to at least $1 - x \ge x$; a nonnegative value does not wrap through one, outside null boundaries. Therefore the third inclusion is equivalent to $0 \le 2V - U < x$. For each $U \in [0, x]$, the allowed interval

$$U/2 \le V < (U + x)/2$$

lies in $[0, x]$ and has length $x/2$. Its area is

$$q(x) = x^2/2, \qquad 0 \le x \le 1/2.$$

Let N_x count the three indexed points in $[0, x)$, and let $P_k(x) = \mathbb{P}(N_x = k)$. Pairwise independence gives $\mathbb{E}N_x = 3x$ and $\mathbb{E}\binom{N_x}{2} = 3x^2$. Together with $P_3 = q$ and total probability one, this determines

$$(P_0, P_1, P_2, P_3) = (1 - 3x + 3x^2 - q,\ 3x - 6x^2 + 3q,\ 3x^2 - 3q,\ q).$$

Consequently, before substituting q,

$$F(x) := \mathbb{E}(N_x/3 - x)^4 = x^4 - \frac{2}{3}x^3 + \frac{2}{27}x^2 + \frac{x}{27} + \frac{4}{9}(1 - 2x)q(x). \tag{1}$$

On the lower half interval this becomes

$$F(x) = x^4 - \frac{10}{9}x^3 + \frac{8}{27}x^2 + \frac{x}{27}.$$

The parameter transformation $(Z, \Delta) \mapsto (-Z, -\Delta)$ modulo one preserves their product uniform law and reflects every point. Outside null endpoint events, it exchanges N_x with $3 - N_{1-x}$. Thus $F(1 - x) = F(x)$. Bounded measurability permits Tonelli's theorem, and

$$\mathbb{E}\,\mathrm{disc}_4(M_3^{Z,\Delta})^4 = 2\int_0^{1/2} F(x)\,dx = 2\left(\frac{1}{160} - \frac{5}{288} + \frac{1}{81} + \frac{1}{216}\right) = \frac{19}{1620}.$$

For an independent triple, the same identity (1) holds with $q_{\mathrm{iid}}(x) = x^3$. On $[0, 1/2]$ its difference from the lattice integrand is therefore

$$F(x) - F_{\mathrm{iid}}(x) = \frac{4}{9}(1-2x)(x^2/2 - x^3) = \frac{2}{9}x^2(1-2x)^2.$$

Reflection applies to both ensembles, so the strict integrated excess is

$$2\int_0^{1/2} \frac{2}{9}x^2(1-2x)^2\,dx = \frac{1}{540}.$$

This also verifies the sign without subtracting approximate values.

For completeness, the iid fourth moment at every point count has a short formula. For independent Bernoulli(x) variables B_i, set $Y_i = B_i - x$ and $a = x(1-x)$. Then $\mathbb{E}Y_i^2 = a$ and $\mathbb{E}Y_i^4 = a - 3a^2$. Only the single-index and paired-index terms survive in the fourth power of their centered sum. Hence

$$\mathbb{E}\left(\frac{1}{n}\sum_i Y_i\right)^4 = \frac{n(a-3a^2) + 3n(n-1)a^2}{n^4}.$$

Integrating a and a^2 gives $1/6$ and $1/30$, respectively, yielding

$$\mathbb{E}\,\mathrm{disc}_4(M_n^{\mathrm{iid}})^4 = \frac{3n-1}{30n^3}. \tag{2}$$

At $n = 3$ this is $4/405$. The whole-law claim for $n \le 2$ follows from the uniform marginal and the independent pair already proved, coordinate by coordinate. This completes Theorem 1.

3 A second counterexample from empty tails

Proposition 2. *For $n \ge 3$ in dimension one,*

$$\mathbb{E}\,\mathrm{disc}_4(M_n^{Z,\Delta})^4 \ge \frac{2}{105(n-1)}.$$

For every integer $n \ge 4$ this lower bound exceeds the iid fourth moment.

Proof. On the open triangle

$$0 < Z < 1/(n-1), \qquad 0 < \Delta < 1 - (n-1)Z,$$

all points lie without wraparound between Δ and $\Delta + (n-1)Z$. Below the first point the empirical count is zero; above the last it is n. The two outer intervals alone contribute

$$\frac{\Delta^5 + (1 - \Delta - (n-1)Z)^5}{5}$$

to the fourth discrepancy moment. Integrating this contribution against the original product density one, first in Δ and then in Z, gives

$$\int_0^{1/(n-1)} \frac{(1-(n-1)Z)^6}{15}\,dZ = \frac{1}{105(n-1)}.$$

The transformation $(Z, \Delta) \mapsto (1 - Z, 1 - \Delta)$ gives a descending no-wrap triangle with the same contribution. Their interiors are disjoint for $n \ge 3$, since their generator intervals are disjoint; boundaries have area zero. Everything omitted is nonnegative, giving the bound.

At $n = 4$ it gives $2/315 > 11/1920$ with margin $5/8064$. In general, by (2), the comparison reduces to $4n^3 - 21n^2 + 28n - 7 > 0$. Writing $n = 4 + t$ makes its left side $25 + 52t + 27t^2 + 4t^3 > 0$ for $t \ge 0$. □

The lower bound alone is insufficient at $n = 3$, where $1/105 < 4/405$. The exact three-point calculation is separate. The rare short progressions explain how a fourth-moment reversal can coexist with pairwise independence.

4 Consistency and verification

For any n, d, an anchored box of volume $v = \prod_r x_r$ has pairwise independent indicator variables in the lattice ensemble. Its normalized count variance is $v(1-v)/n$, as for iid points. Thus

$$\mathbb{E}\,\mathrm{disc}_2(M_n^{Z,\Delta})^2 = \frac{2^{-d} - 3^{-d}}{n} = \mathbb{E}\,\mathrm{disc}_2(M_n^{\mathrm{iid}})^2.$$

Indexing and the weight $1/n$ are retained even at collisions. Parameters causing collisions or tested endpoint equalities are null sets; they are not conditioned away. Closed parameter cubes and half-open cubes give the same expectations.

The result concerns $\mathbb{E}\,\mathrm{disc}_p^p$. It does not order $\mathbb{E}\,\mathrm{disc}_p$, settle the separate existence question for good lattice generators, or refute discrepancy estimates restricted to selected generators.

The source archive includes the unchanged source-PR documents, three new independent mathematical audits, current priority and chronology reviews, and exact rational standard-library verification programs. Besides the inclusion calculation, a separate complete affine-cell integration gives $19/1620$ for three points and the supplementary value $139/17280$ for four points. The latter value is not needed by either analytic counterexample. Ordinary and optimized Python runs agree. The archive documents source versions and inaccessible literature; successful checks do not establish exhaustive priority or formal proof certification.

AI assistance and review status. AI tools were used extensively in proof development, exact computation, adversarial verification, literature research and manuscript preparation. This note and its supporting materials are unrefereed. Independent AI adversarial reviews are not external human peer review, journal acceptance or proof-assistant certification.

References

[1] E. Novak (joint work with A. Hinrichs), *New bounds for the star discrepancy*, in Discrepancy Theory and Its Applications, Oberwolfach Report 13/2004, pp. 696–699. doi:10.4171/OWR/2004/13.

[2] A. Ferudun, *A Negative Answer to a Question of Novak on Randomly Shifted Lattices and the Expected p-th Power of the L_p Discrepancy*, EulerSolve Research Papers, OWR-723-005, version 1.0, 14 October 2026, unrefereed preprint. doi:10.5281/zenodo.23372668.

[3] A. Kriebel, source proof and audit for problem 30000052, Math repository PR 1135, GitHub creation record 10 October 2026, reviewed head `e809a9c40f4ccdaced6f45f10b1dc8b7725a2119`. Exact source identities and dated API receipts accompany this note.