

Three anti-invariant classes on a nonintegrable four-manifold

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Abstract

We construct smooth nonintegrable almost complex structures on the product of a genus-two curve and an elliptic curve with at least three independent real anti-invariant de Rham classes. The structures converge smoothly to the product complex structure and equal it outside one coordinate ball. The construction gives a negative answer to the unrestricted integrability conjecture of Draghici–Li–Zhang. The separate generic-vanishing statement is a prior theorem of Tan–Wang–Zhang–Zhu.

1 Statement and context

For an almost complex structure J on a closed smooth four-manifold M , set

$$H_J^- = \{[\alpha] \in H_{\mathrm{dR}}^2(M; \mathbb{R}) : d\alpha = 0, \alpha(JX, JY) = -\alpha(X, Y)\}, \quad h_J^- = \dim_{\mathbb{R}} H_J^-.$$

These groups and their cohomological setting are due to Li–Zhang and Draghici–Li–Zhang [1, 2, 3]. An early exact existence question appears in [2, Question 3.27]. Conjecture 2.5 of [3] asserts that $h_J^- \geq 3$ forces integrability. It is repeated in Tomassini’s contribution to Report 33/2020 [4, p. 1687]; the report’s bibliographic year is 2020 and its EMS publication date is September 14, 2021.

Theorem 1. *Let Σ be the smooth compact completion of $y^2 = z^6 - 1$, let $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, and let J_0 be the product complex structure on $M = \Sigma \times E$. There exist $\varepsilon_0 > 0$ and a smooth family J_ε , $|\varepsilon| < \varepsilon_0$, such that*

1. $J_\varepsilon = J_0$ outside a fixed coordinate ball and $J_\varepsilon \rightarrow J_0$ in C^∞ ;
2. three fixed linearly independent classes in $H_{\mathrm{dR}}^2(M; \mathbb{R})$ have closed J_ε -anti-invariant representatives;
3. J_ε is nonintegrable whenever $\varepsilon \neq 0$.

In particular, $h_{J_\varepsilon}^- \geq 3$ for $\varepsilon \neq 0$.

The other conjecture in the source concerns generic vanishing. Tan, Wang, Zhang and Zhu proved that $h_J^- = 0$ is open and dense in the C^∞ topology on all smooth almost complex structures on a closed four-manifold [5, Theorem 1.1]. The special family above does not contradict that theorem. We compute a lower bound, not the exact value of h_J^- .

2 Construction and proof

The polynomial $z^6 - 1$ has six simple roots. Its compact double cover of $\mathbb{CP}^1$ is connected and has genus two by Riemann–Hurwitz. The differentials

$$\eta_0 = \frac{dz}{y}, \quad \eta_1 = \frac{z dz}{y}$$

extend holomorphically and are linearly independent. At a finite branch point, y is a coordinate and $dz/y = dy/(3z^5)$ has no pole. At either point over infinity put $s = 1/z$, $q = y/z^3$; then $q^2 = 1 - s^6$ and

$$\eta_0 = -\frac{s ds}{q}, \quad \eta_1 = -\frac{ds}{q}.$$

Let dw denote the global holomorphic one-form on E , and define

$$\Omega_j = \eta_j \wedge dw, \quad \alpha = \operatorname{Re} \Omega_0, \quad \beta = \operatorname{Im} \Omega_0, \quad \gamma = \operatorname{Re} \Omega_1.$$

These are closed and J_0 -anti-invariant.

Their real cohomology classes are independent. Indeed, for real a, b, c , put $H = (a - ib)\Omega_0 + c\Omega_1$ and $F = \operatorname{Re} H$. Then $F = a\alpha + b\beta + c\gamma$ and $F^2 = \frac{1}{2}H \wedge \overline{H}$ is nonnegative in the complex orientation, and positive wherever $H \neq 0$. If F were exact, Stokes' theorem would give $\int_M F^2 = 0$, so $H = 0$. Independence of η_0, η_1 gives $a = b = c = 0$.

The simultaneous local model. Near $(z, y, w) = (0, i, 0)$ choose the holomorphic branch $y = y(z)$ and a local lift of w . The coordinates

$$\zeta = z, \quad \xi = \frac{w}{y(z)}$$

have nonzero Jacobian and satisfy $\Omega_0 = d\zeta \wedge d\xi$ and $\Omega_1 = \zeta \Omega_0$. Writing $\zeta = u + iv$ and $\xi = x + it$, we obtain

$$\alpha = du \wedge dx - dv \wedge dt, \quad \beta = du \wedge dt + dv \wedge dx, \quad \gamma = u\alpha - v\beta. \quad (1)$$

Here t is a coordinate, and ε is the deformation parameter. Choose a relatively compact coordinate ball B about the origin and a smooth real cutoff $\rho \in C_c^\infty(B)$ equal to one near the origin. Set

$$R = \frac{\varepsilon}{2}\rho x^2.$$

The one-forms $R dv$ and $Rv dv$ extend smoothly by zero to M . Define global closed forms

$$\alpha_\varepsilon = \alpha, \quad \beta_\varepsilon = \beta + d(R dv), \quad \gamma_\varepsilon = \gamma - d(Rv dv). \quad (2)$$

Their cohomology classes are the three original independent classes. The identity $d(Rv dv) = v d(R dv)$ holds for the full cutoff, not only where $\rho = 1$. Consequently,

$$\gamma_\varepsilon = u\alpha - v\beta_\varepsilon \quad \text{on } B. \quad (3)$$

This simultaneous retention of the third closed class is the key relation.

Realizing the anti-invariant plane. Put $\text{vol} = du \wedge dv \wedge dx \wedge dt$. Subscripts on R denote partial derivatives. Direct expansion gives

$$\beta_\varepsilon = du \wedge dt + (1 - R_x)dv \wedge dx - R_t dv \wedge dt + R_u du \wedge dv,$$

and hence

$$\alpha^2 = 2\text{vol}, \quad \alpha \wedge \beta_\varepsilon = R_t \text{vol}, \quad \beta_\varepsilon^2 = 2(1 - R_x)\text{vol}. \quad (4)$$

For sufficiently small $|\varepsilon|$ we have uniformly $|R_x| \leq 1/4$, $|R_t| \leq 1$; these bounds follow because ρ is fixed and compactly supported. Thus

$$s = \frac{R_t}{2}, \quad D = 1 - R_x - \frac{R_t^2}{4} \geq \frac{1}{2}.$$

Define on B

$$B_\varepsilon = \frac{\beta_\varepsilon - s\alpha}{\sqrt{D}}, \quad \Psi_\varepsilon = \alpha + iB_\varepsilon.$$

Equation (4) implies

$$\Psi_\varepsilon^2 = 0, \quad \Psi_\varepsilon \wedge \overline{\Psi_\varepsilon} = 4\text{vol}. \quad (5)$$

In a complex vector space of dimension four a nonzero two-form with square zero has rank two and is decomposable. Its contraction kernel therefore has complex dimension two. The second identity in (5) says that this kernel and its conjugate are complementary. Declaring

$$T_{J_\varepsilon}^{0,1} = \{Z \in TM \otimes \mathbb{C} : \iota_Z \Psi_\varepsilon = 0\}$$

therefore defines a smooth real almost complex structure on B . It agrees with J_0 at $\varepsilon = 0$. This is the usual positive two-plane description of an almost complex structure in dimension four [6]; the preceding kernel argument supplies the needed realization directly.

On a collar of ∂B , all jets of R vanish, so that $s = 0$, $D = 1$, $\Psi_\varepsilon = \Omega_0$ and $J_\varepsilon = J_0$. Extend J_ε by J_0 outside B . The extension is smooth. The fixed compact support and the uniform positive denominator imply $J_\varepsilon \rightarrow J_0$ in every finite smooth norm, hence in C^∞ . Zeros of Ω_0 elsewhere on M cause no difficulty: the kernel construction is used only on B , where Ω_0 is nonzero.

On B , α and β_ε are real combinations of the real and imaginary parts of Ψ_ε , so they are J_ε -anti-invariant. Equation (3) gives the same conclusion for γ_ε . Outside B they are the original J_0 -anti-invariant forms. Equation (2) now proves the claimed three independent classes.

Nonintegrability. Where $\rho = 1$ put $k = 1 - \varepsilon x > 0$ and $k' = \partial_x k = -\varepsilon$. There

$$\Psi_\varepsilon = (du + i\sqrt{k}dv) \wedge (dx + ik^{-1/2}dt) = \phi_1 \wedge \phi_2.$$

Thus (ϕ_1, ϕ_2) is a $(1, 0)$ coframe, and

$$d\phi_1 = \frac{ik'}{2\sqrt{k}} dx \wedge dv, \quad d\phi_1 \wedge \phi_1 \wedge \phi_2 = \frac{k'}{2k} \text{vol}. \quad (6)$$

The last expression is nonzero for $\varepsilon \neq 0$. For an integrable structure, the differential of a $(1, 0)$ form has no $(0, 2)$ component, so its wedge with a $(2, 0)$ form vanishes in complex dimension two. Equation (6) proves nonintegrability and completes the proof of Theorem 1.

3 Comparison and limits

The two-plane realization and closed-pair framework are established ingredients [6, 7]; the third-class identity (3) is the specific step used here. Three independent cohomology classes do not mean three pointwise independent two-forms: on B their representatives span a real plane, with the variable coefficients in (3).

Two distinctions explain why familiar integrability criteria do not obstruct this construction. First, on the core of B ,

$$B_\varepsilon = k^{-1/2} du \wedge dt + \sqrt{k} dv \wedge dx, \quad dB_\varepsilon = -\frac{k'}{2k^{3/2}} dx \wedge du \wedge dt \neq 0.$$

A closed real pair need not remain closed after variable normalization. In particular, Ψ_ε is not a closed complex symplectic form; the closed normalized-pair criterion [7, Theorem 6] is inapplicable. Second, on this core $J_0 \partial_u = \partial_v$ whereas $J_\varepsilon \partial_u = k^{-1/2} \partial_v$. A positive metric compatible with both would give $g_{uu} = g_{vv} = k^{-1} g_{vv}$, impossible where $k \neq 1$. Thus no common compatible metric with this product J_0 exists. No compatibility with a fixed symplectic form, spatial analyticity or exact value of $h_{J_\varepsilon}^-$ is asserted.

Earlier compact four-dimensional nonintegrable examples attain $h_J^- = 2$ [3, 8]. Hind–Tomassini’s final arXiv version [8, Proposition 4.2] distinguishes this compact case from its noncompact and real-six-dimensional larger examples; localized deformation alone is consequently not a new ingredient. Chen–Zhang’s accepted manuscripts, I, §6.3 [9], and II, §5.1 [10], give closely related nonintegrable products $T^2 \times S$. Their computed plurigenera are different from the real closed anti-invariant cohomology used here, and II explicitly leaves the latter calculation for those models as an interesting question. We do not claim that no further argument about those older models could supply such classes.

A bounded literature audit through October 16, 2026 found no verified earlier qualifying counterexample. It does not establish firstness, novelty, present openness or exhaustive absence. The supplementary audit records actual reading and access limits: some journal-final comparisons were unavailable; Chen–Zhang accepted versions were read; only selected author-uploaded passages of the 2012 DLZ survey were accessible; and Geiges’s 1996 paper remained a bibliographic lead. Withdrawn claims [11, 12] are not used as theorems. These historical limits do not enter the explicit proof.

Verification and disclosures. AI tools were used extensively in the construction, drafting, literature search, symbolic computations and adversarial checking. This research note is unrefereed; no independent human peer review or formal verification is claimed. The accompanying source and verification package preserves the submitted proof and checks, independent audit reports and reproducible finite algebra. Its integrity and computation checks do not replace the global argument or certify historical priority. The author is responsible for the content. This note is released under CC BY 4.0.

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