Tail generation for symmetric free-product states: a general deduction

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Abstract

For every unital C^*-algebra and every state on its countable universal unital free product invariant under finite permutations, the recursive tail C^*-algebra of Dykema–Köstler–Williams generates the tail von Neumann algebra in the original GNS representation. More precisely, the canonical expectation of the represented free-product algebra is weak-operator dense in the tail with the exact norm bound of each target. This is a short general deduction from the earlier coefficient-stabilization method and Kaplansky density, answering the explicit generation question. No faithfulness or separability is required. We explain the close relation to Liu's bilateral future-expectation argument and retain bounded priority and source-version qualifications.

1 The question, its definition, and the conclusion

Let A be a unital C^*-algebra, let $\mathcal{A} = *_{j\in\mathbb{N}}A$ be its universal unital free product, with $\mathbb{N} = \{1, 2, \ldots\}$, and let ψ be a state invariant under every finite permutation of the copies. Write (π, H, ξ) for its GNS representation, $\pi_j : A \to B(H)$ for the represented j-th copy, and

$$M = \pi(\mathcal{A})'', \qquad \mathcal{C}_N = C^*(\pi_j(A) : j > N), \qquad \mathcal{T} = \bigcap_{N\geq 1} \mathcal{C}_N'', \qquad Q = C^*(\mathcal{T} \cup \pi(\mathcal{A})).$$

All closures and commutants here are in $B(H)$, with its original cyclic representation; C^*-algebras generated by unital copies contain the common unit.

We use the canonical expectation $F : Q \to \mathcal{T}$ of Dykema–Köstler–Williams (DKW), Theorem 5.1.10(iv) [1]. The source provides an isometric shift α on Q, satisfying $\alpha(\pi_j(a)) = \pi_{j+1}(a)$, and

$$F(x) = \mathrm{WOT}-\lim_{k\to\infty} \alpha^k(x) \quad (x \in Q). \tag{1}$$

It also provides spatial finite-permutation actions β_σ with implementing unitaries fixing ξ. These are results for general symmetric states in DKW §5, rather than assumptions of quantum symmetry. In particular F is contractive and fixes $\mathcal{T}$; a normal extension to M is not asserted.

Precisely as in DKW Definition 5.3.1, $\mathcal{D}$ is the least unital C^*-subalgebra of $\mathcal{T}$ such that, for every $m \geq 1$, every $a_1, \ldots, a_m \in A$, every $i_1, \ldots, i_m \in \mathbb{N}$, and every $d_1, \ldots, d_{m-1} \in \mathcal{D}$,

$$F\big(\pi_{i_1}(a_1)d_1\pi_{i_2}(a_2)\cdots d_{m-1}\pi_{i_m}(a_m)\big) \in \mathcal{D}.$$

There is no distinct-index restriction. The family is nonempty because $\mathcal{T}$ has this property, and its intersection defines $\mathcal{D}$.

Theorem 1. *For every A, ψ as above, $\mathcal{D}'' = \mathcal{T}$. In fact, for each $t \in \mathcal{T}$ there is a net $x_\lambda \in \pi(\mathcal{A})$ such that*

$$\|x_\lambda\| \le \|t\|, \qquad \|F(x_\lambda)\| \le \|t\|, \qquad F(x_\lambda) \longrightarrow t \quad \textit{in WOT}.$$

Consequently $\overline{F(\pi(\mathcal{A}))}^{\mathrm{WOT}} = \mathcal{T}$.

This answers DKW Question 5.3.5 and the separately posed Question 2 in the 2015 and 2016 Oberwolfach reports [4, 5]. Their other question, whether $\mathcal{D} \subseteq \pi(\mathcal{A})$, is separate and is not answered here. DKW Theorem 7.3 already proves tail generation for quantum symmetric states, including nonfaithful states. The present statement imposes no faithfulness of ψ or π, separability, trace, factoriality, or quantum symmetry.

2 Future compression and bounded approximation

Put

$$\mathcal{B}_N = C^*(\pi_j(A) : 1 \le j \le N), \qquad H_N = \overline{\mathcal{B}_N \xi}, \qquad P_N : H \to H_N$$

with P_N the orthogonal projection.

Lemma 2. *For every $N \ge 1$ and $x \in \mathcal{C}_N$,*

$$P_N F(x) P_N = P_N x P_N.$$

Proof. First take a finite-copy polynomial x whose indices belong to a finite set $S \subset \{N+1, N+2, \dots\}$. Fix $k \ge 1$. The injection $i \mapsto i + k$ from S onto $S + k$ extends to a permutation of the finite set $U = S \cup (S + k)$: complete it by any bijection $U \setminus S \to U \setminus (S + k)$. Extend that permutation by the identity off U. It fixes every first-N copy and transports every occurrence of each index simultaneously. Overlap between S and $S + k$, repeated indices, and shared indices among different monomials cause no problem.

For $y, z \in \mathcal{B}_N$, state invariance gives, for every k,

$$\langle \alpha^k(x) y\xi, z\xi \rangle = \psi(\beta_\sigma(z^* x y)) = \psi(z^* x y) = \langle xy\xi, z\xi \rangle.$$

The notation ψ on represented elements means its GNS vector state. The permutation action is well-defined even in a nonfaithful GNS representation: state preservation preserves the GNS null space and the GNS inner product, and its unitary fixes ξ. Taking $k \to \infty$ uses (1) only for this fixed input x. Future polynomials are norm dense in $\mathcal{C}_N$; contractivity of F extends the identity to every $x \in \mathcal{C}_N$. Finally $\mathcal{B}_N \xi$ is dense in H_N, and boundedness of $F(x)$ and x extends it to all vector pairs in H_N. □

We recall exactly the norm-ball form of Kaplansky density used below [3]. If $C \subset B(H)$ is a unital concrete C^*-algebra and $t \in C''$, then its ball of radius $\|t\|$ is strongly, hence weak-operator, dense at t. For completeness, the reduction of the nonselfadjoint case to the usual selfadjoint theorem preserves this exact radius. Apply selfadjoint norm-ball Kaplansky in $M_2(C)$ to

$$Y = \begin{pmatrix} 0 & t \\ t^* & 0 \end{pmatrix} \in M_2(C''), \qquad \|Y\| = \|t\|.$$

The upper-right corners of selfadjoint approximants of norm at most $\|t\|$ lie in C, have that same norm bound, and converge strongly to t, by testing vectors $(0, \eta)$. Thus no factor two or norm-attaining lift from an abstract quotient is needed. The matrix device already appears in Kaplansky's proof, which credits P. R. Halmos.

Proof of Theorem 1. Fix $t \in \mathcal{T}$, and set $r = \|t\|$. If $r = 0$, take the constant zero net. The increasing union $\bigcup_N H_N$ is dense in H: every finite sum of copywords uses only finitely many indices, and ξ is cyclic for $\pi(\mathcal{A})$. This does not require a countable dense subset of A or H. Hence $P_N \to I$ strongly.

Specify a WOT neighborhood of t by finitely many vector pairs (η_l, ζ_l), $1 \le l \le m$, and an error $\varepsilon > 0$. Choose a single N so large that, for all l,

$$2r\big(\|\eta_l - P_N\eta_l\|\|\zeta_l\| + \|P_N\eta_l\|\|\zeta_l - P_N\zeta_l\|\big) < \varepsilon/2. \tag{2}$$

Since $t \in \mathcal{C}_N''$, the preceding norm-ball theorem gives $x \in \mathcal{C}_N$, $\|x\| \le r$, with

$$|\langle (x-t)P_N\eta_l, P_N\zeta_l\rangle| < \varepsilon/2 \quad (1 \le l \le m).$$

The algebra $\mathcal{C}_N$ is contained in $\pi(\mathcal{A})$, since the image of a C^*-homomorphism is norm closed and contains every copy image.

Put $d = F(x)$. Contractivity gives $\|d\| \le r$ and $\|d - t\| \le 2r$. Lemma 2 makes the compressed coefficient of $d - t$ equal to the displayed coefficient of $x - t$. Splitting the original vectors at P_N therefore gives

$$\begin{aligned}|\langle (d-t)\eta_l, \zeta_l\rangle| \le\ & |\langle (d-t)P_N\eta_l, P_N\zeta_l\rangle| \\ & + 2r\big(\|\eta_l - P_N\eta_l\|\|\zeta_l\| + \|P_N\eta_l\|\|\zeta_l - P_N\zeta_l\|\big) < \varepsilon.\end{aligned}$$

Directing WOT neighborhoods by reverse inclusion gives the required bounded net of represented inputs. Conversely its outputs lie in the WOT-closed algebra $\mathcal{T}$.

By the definition of $\mathcal{D}$, setting every inserted $d_j = 1$ shows that F of each finite copyword belongs to $\mathcal{D}$. Their linear span is norm dense in $\pi(\mathcal{A})$. Boundedness of F and norm closure of $\mathcal{D}$ imply $F(\pi(\mathcal{A})) \subseteq \mathcal{D} \subseteq \mathcal{T}$. The established density now gives $\mathcal{T} \subseteq \mathcal{D}'' \subseteq \mathcal{T}$, as required. □

3 What the argument uses, and its relation to prior work

The proof never passes a WOT-convergent input net through F. For each chosen future algebra it uses an exact compression identity, then controls the remaining vector coefficients by a uniform norm bound. The uniform bound is essential; convergence on a dense vector subspace alone need not imply WOT convergence.

Nonnormality is a real obstruction to a different shortcut. In the compact-plus-scalar example recorded by DKW, Example 5.2.1 (credited there to Liu), take $H = \ell^2(\{0, 1, \ldots\})$, $a_j = e_{0j} + e_{j0}$, $Q = K(H) + \mathbb{C}I$, and $\mathcal{T} = \mathbb{C}e_{00} + \mathbb{C}(I - e_{00})$. The canonical expectation is

$$F(\lambda I + K) = \lambda I + K_{00}e_{00}.$$

The projections $q_m = \sum_{j=0}^{m} e_{jj}$ increase strongly to I, but $F(q_m) = e_{00} \ne F(I) = I$, excluding a normal extension fixing these values. Nevertheless $F(a_j^2) = e_{00}$, so ordinary moments already span the tail in this example, as the theorem predicts.

Attribution and the narrow contribution. DKW Lemma 5.1.8, the proof of Theorem 5.1.10, and Remark 5.1.11 already contain finite-word coefficient stabilization and bounded-vector extension. Liu's bilateral treatment [2, §6.2], especially Lemma 6.8 and Proposition 6.10 in the inspected arXiv version, gives a closely related past/future identity and normality of the expectation on each fixed future von Neumann algebra. Neither a new coefficient principle nor a new Kaplansky method is claimed here. The contribution is the short general deduction for the explicit DKW recursive C^*-algebra in its original representation.

The relation to Liu is closer than a superficial distinction between bilateral and unilateral indexing suggests. Extend the symmetric law to indices $\mathbb{Z}$; in its bilateral GNS representation the shift is spatial and normal. The same prior stabilization argument, extended to blocks of elements of arbitrary A and bounded nets, yields a normal expectation from each fixed future algebra onto its positive tail. Restriction to the positive cyclic subspace gives a normal surjection $\rho : M_+ \to M$. Its image of the bilateral positive tail is exactly $\mathcal{T}$: for $t \in \mathcal{T}$, the norm-$\|t\|$ lifts in the successive future algebras form nonempty nested ultraweakly compact sets, and their intersection provides a tail lift. Compatibility of the shift limits on C^*-inputs then gives the same density conclusion. One must prove this tail-lifting step; images cannot simply be commuted with intersections. Nor need the bilateral expectation descend through ρ on all of M_+. The accompanying `RELATION_TO_PRIOR_WORK.md` supplies this bridge in detail. It emphasizes the corollary character of the result, rather than asserting that Liu stated the full arbitrary-A DKW answer.

Versions, priority limits, and provenance. The relevant DKW §5 and Liu §6.2 bodies were read in the current responses at the versioned arXiv URLs below. The DKW response has internal running headers dated August 2021 although arXiv labels v3 as September 2014; no identity with historical 2014 or journal bytes is inferred. Both AMS full-body routes were inaccessible, and Liu's dissertation body was not recovered. A bounded primary-source investigation found no authenticated earlier publication of the full general answer. It does not certify exclusive firstness, worldwide novelty, or continuing openness. The proof was already public in project PR1329 (13 October 2026); this note is dated 22 October 2026 and is not backdated. Catalog numbers 30003208 and30002897 name the same generation question.

Verification, authorship, and status. The source archive includes portable standard-library exact controls and deliberately false variants for the finite permutation and compact-plus-scalar mechanisms. These finite checks supplement the written proof and do not certify arbitrary C^*-algebras or priority. AI tools were extensively involved in research planning, mathematical exploration, proof development and reconstruction, computation, source and literature investigation, writing, editing, and adversarial verification. Alec Kriebel is the named human author. This work is unrefereed, without conventional independent human peer review, journal acceptance, or formal proof-assistant certification; internal AI audits do not supply those certifications. Prose is licensed CC BY 4.0; accompanying authored code is MIT licensed.

References

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