A supercritical nonuniqueness example on a graph roughly isometric to the square lattice

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Abstract

We give an explicit deterministic graph of maximum degree four, roughly isometric to the square lattice, on which iid bond percolation has at least two infinite clusters with probability greater than one third at parameter 19/20, while its critical probability is at most 9/10. Edges internal to two opposite quadrants are retained; every other lattice edge is replaced by a path of length 100. Elementary dual parity and all-distance path bounds prove the assertions without a lattice threshold formula or a transitivity assumption. The example refutes the explicitly declared all-supercritical interpretation of Benjamini's 2013 Problem 9.6. The original question omits its probability quantifier; no unique author intent, exclusive priority, or current open status is asserted.

1 Statement and the source question

All graphs here are countable, simple, undirected and connected, with graph distance. In Bernoulli bond percolation every edge is independently retained with the same probability p. Write N_∞ for the number of infinite open components and

$$p_c(G, o) = \inf\{p : \mathbb{P}_p(o \leftrightarrow \infty) > 0\}.$$

For a connected graph, positive survival at one vertex implies positive survival at every vertex at the same $p > 0$: force open a fixed finite connecting path. Indeed, conditioning those finitely many bonds to be open can only increase any increasing event, by coupling the other bonds identically. Hence $\mathbb{P}_p(o \leftrightarrow \infty) \geq p^{|F|}\mathbb{P}_p(v \leftrightarrow \infty)$ for such a path F from o to v. We may therefore write $p_c(G)$.

We use precisely the multiplicative-plus-additive rough-isometry convention of [1, Definition 1.25, manuscript p. 9]: a map $f : V(G) \to V(H)$ and a constant $C > 0$ satisfy

$$C^{-1}d_H(f(x), f(y)) - C \leq d_G(x, y) \leq Cd_H(f(x), f(y)) + C,$$

and every vertex of H is at distance strictly less than C from the range.

Problem 9.6 of the 30 October 2013 author manuscript [1, p. 64] asks for uniqueness on bounded-degree graphs roughly isometric to $\mathbb{Z}^d$. Its sentence supplies no p-quantifier. The chapter's preceding page explicitly discusses $p > p_c$, while transitivity is a separate hypothesis of nearby theorems. We address the following *declared interpretation*: for every such graph and every fixed $p_c(G) < p \leq 1$, $\mathbb{P}_p(N_\infty = 1) = 1$. A negative example in dimension two refutes this interpretation. It does not refute a weaker assertion of uniqueness only sufficiently near one, determine the author's unexpressed intent, or require a simultaneous assertion for all p in one coupling.

Theorem 1. There is a deterministic graph G of maximum degree four, roughly isometric to $\mathbb{Z}^2$ with $C = 100$, such that

$$p_c(G) \le \frac{9}{10} < \frac{19}{20}, \qquad \mathbb{P}_{19/20}(N_\infty \ge 2) > \frac{2200741829}{6461571200} > \frac{1}{3}.$$

At $p = 9/10$ every vertex has positive probability of an infinite open cluster.

2 The graph and the exact coarse model

In the nearest-neighbor square lattice let

$$A = \{(x, y) : x \ge 1,\ y \ge 1\}, \qquad B = \{(x, y) : x \le -1,\ y \le -1\}.$$

Call edges internal to A or internal to B strong; all others are weak. Keep each strong edge, and replace every weak edge by a path of exactly 100 edges with 99 new internal vertices private to that path. There are no other edges. Original vertices have degree four; new vertices have degree two. The resulting graph is infinite, countable, connected, simple and deterministic. The subdivision bound is global, not a length growing with location.

Fix original vertices under $f : G \to \mathbb{Z}^2$, and send an internal path vertex to its nearer original endpoint, choosing ties consistently. Adjacent vertices have equal or neighboring images, so $d_{\mathbb{Z}^2}(f(x), f(y)) \le d_G(x, y)$. Each vertex is within 50 actual edges of its image, and a lattice geodesic between original vertices expands to a path at most 100 times as long. Thus, for all vertices,

$$d_{\mathbb{Z}^2}(f(x), f(y)) \le d_G(x, y) \le 100 d_{\mathbb{Z}^2}(f(x), f(y)) + 100.$$

Surjectivity of f supplies the strict 100-net condition. These inequalities imply the displayed source convention with the single constant 100, including for chain midpoints.

At actual bond parameter p, declare a coarse strong edge open when its single bond is open, and a coarse weak edge open when *all* its 100 bonds are open. Coarse states are independent with probabilities p and $q = p^{100}$, respectively: their actual bond sets are disjoint. Sharing endpoints creates no dependence in a bond model. This familiar series reduction is used, for example, in [7, Section 2]; it is not a new probabilistic mechanism.

Actual and coarse connectivity of original vertices agree. A simple actual path between originals must traverse every chain it enters completely, since its internal vertices have degree two. Conversely, expand open coarse edges. An actual component containing no original lies inside one finite chain. A component containing finitely many originals lies in those vertices and their finitely many incident finite chains, and is finite. Hence every infinite component contains infinitely many originals. Restriction to originals and expansion therefore give a bijection between actual and coarse infinite components, even when partial chain branches are present. The inhomogeneous coarse description is exactly the requested homogeneous iid model on G.

3 Survival inside a strong quadrant

Let Q be the nearest-neighbor graph on $\mathbb{N}_0^2$. We prove the explicit bound

$$\mathbb{P}^Q_{9/10}((0, 0) \leftrightarrow \infty) \ge \frac{143}{245}. \tag{1}$$

For a finite set $D \subset \mathbb{N}_0^2$ containing the origin, draw the dual edges crossing primal Q-edges with exactly one endpoint in D. Interior dual vertices are square centers $(i + 1/2, j + 1/2)$. Boundary dual endpoints are $(i + 1/2, -1/2)$ and $(-1/2, j + 1/2)$, where $i, j \ge 0$.

At an interior dual vertex the degree is 0, 2 or 4, since the cyclic binary membership sequence around its square has an even number of changes. A bottom or left boundary vertex has degree at most one. On the primal bottom ray, membership starts at one and is eventually zero, so the number of bottom degree-one endpoints is odd. Some finite dual component consequently has an odd number of bottom endpoints. By the handshaking lemma it also has an odd, and hence positive, number of left endpoints. A simple path in that component joins a bottom endpoint $(i + 1/2, -1/2)$ to a left endpoint $(-1/2, j + 1/2)$. Boundary vertices cannot be internal because they have degree one. Its length m satisfies

$$m \geq i + j + 2 \geq 2.$$

The argument allows holes, disconnected D, diagonal contacts and degree-four dual vertices; no choice of an outer-boundary tour is required.

If the open corner cluster is finite, take it as D. Every crossed primal edge is closed. For each length $m \geq 2$ there are at most m possible bottom starts, since $i \leq m - 2$. From one start there are at most $4 \cdot 3^{m-1}$ self-avoiding dual paths of length m. Each eligible path crosses m distinct actual Q-edges, so its probability of being closed is $(1 - t)^m$. Ineligible paths enlarge only the count; no probability is assigned to edges outside Q. For $z = 3(1 - t) < 1$, the union bound gives

$$\mathbb{P}_t^Q(\text{corner cluster finite}) \leq \frac{4}{3} \sum_{m \geq 2} m z^m = \frac{4}{3} \frac{z^2(2 - z)}{(1 - z)^2}. \tag{2}$$

The series identity follows by subtracting its $m = 1$ term from $\sum_{m \geq 1} m z^m = z/(1 - z)^2$. At $t = 9/10$, the last expression is $102/245$, proving (1). In particular $p_c(G) \leq 9/10$, and the finite-path forcing argument gives positive survival at every actual vertex. Coupling edge states by common independent uniforms makes quadrant survival increasing in p. Thus at $p = 19/20$ the events S_A, S_B that the respective corners $(1, 1), (-1, -1)$ survive using only their internal strong edges each have probability at least $143/245$. This uses monotonicity of survival, not monotonicity of uniqueness.

4 A bound on every weak connection

Fix $p = 19/20$ and put $q = p^{100}$, $r = 3q$. If an open coarse path connects A to B, stop at its first B vertex and begin at its last earlier A vertex. The retained segment has all internal vertices outside $A \cup B$, so every edge is weak. Loop erasure preserves these properties. This handles arbitrary detours, distant crossings and earlier strong-edge excursions.

Let W denote existence of any finite self-avoiding all-weak open coarse path from A to B. Such a path starts at $(1, n)$ or $(n, 1)$, $n \geq 1$, and its length m is at least its distance $n + 3$ to B. Double-counting the corner is harmless. There are at most $4 \cdot 3^{m-1}$ candidate self-avoiding paths per start. Each eligible all-weak path has open probability q^m; paths containing strong edges are again used only to enlarge the count. For $r < 1$,

$$\mathbb{P}_p(W) \leq 2 \sum_{n \geq 1} \sum_{m \geq n+3} 4 \cdot 3^{m-1} q^m = \frac{8}{3} \frac{r^4}{(1 - r)^2}. \tag{3}$$

This sums every boundary start and every length. There is no finite-window cutoff or assumed uniform closed barrier.

Bernoulli's inequality gives $(20/19)^{20} \geq 1 + 20/19 > 2$. Raising to the fifth power yields $q < 1/32$, so $r < 3/32$. The positive series in (3) is increasing in r, whence

$$\mathbb{P}_{19/20}(W) < \frac{8}{3} \frac{(3/32)^4}{(1 - 3/32)^2} = \frac{27}{107648}. \tag{4}$$

The events S_A, S_B, W^c depend respectively on the internal-A bonds, internal-B bonds and weak-chain bonds. These are disjoint families. Their generated sigma-fields are independent: independence for cylinder events extends to the sigma-fields by the monotone-class argument. All events are measurable, with survival a countable intersection of finite-distance connection events and W a countable union over finite paths. On $S_A \cap S_B \cap W^c$, the two infinite surviving clusters cannot lie in one actual component. Such a connection would supply a finite actual, hence coarse, A-to-B path and force W. Consequently

$$\mathbb{P}_{19/20}(N_\infty \geq 2) \geq \mathbb{P}(S_A)\mathbb{P}(S_B)\mathbb{P}(W^c) > \left(\frac{143}{245}\right)^2 \frac{107621}{107648} = \frac{2200741829}{6461571200}.$$

Subtracting $1/3$ gives $140654287/19384713600 > 0$. This proves Theorem 1. The parameter is strictly supercritical; positive-probability nonuniqueness suffices to contradict almost-sure uniqueness. No 0–1–infinity law, deterministic cluster count, transitivity, or uniqueness-monotonicity assertion is used.

5 Related work, attribution and limits

Bounded subdivision, series reduction, dual parity, path counting and product independence are established elementary tools. Benjamini and Schramm [2], Example 1 and Proposition 1, already give a planar example with a supercritical nonuniqueness regime. It is an enhanced binary tree roughly isometric to a hyperbolic sector, rather than a Euclidean lattice; its stated planar model is site percolation. Their Lemma 1 concerns uniqueness sufficiently near one. Benjamini–Kozma [3, Theorem 1] give critical nonuniqueness on a subexponential-growth product graph, without the required lattice rough-isometry assertion.

Teixeira [4, Theorem 1.4] and Candellero–Teixeira [5, Theorem 1.8] prove high-density uniqueness under their respective local-isoperimetric/volume or rough-transitivity, growth, isoperimetric and decoupling assumptions, with site-model distinctions. Hutchcroft–Tointon [6, Theorem 1.8] give a high-density bond result under a uniform finite-exhaustion isoperimetric hypothesis, and their Theorem 1.10 has a separate uniform-connection premise. A threshold below one in these theorems is not an identification with p_c. The present example is compatible with a weaker near-one question.

Hilário–Sá–Sanchis–Teixeira [7, Section 2 and Remark 5] explicitly use independent coarse probabilities that are powers of p. Their full-supercritical uniqueness statement concerns an *annealed* random stretched-lattice law, using its translation invariance, ergodicity and finite energy. Those properties of the averaged law do not assert uniqueness for every deterministic angular stretching pattern. No comparison theorem is imported into the proof above.

The primary source question, definition, bond convention and chapter context were read in the pinned 30 October 2013 manuscript. The published chapter has different pagination; byte identity with it is not asserted. Relevant statements of the comparison papers were read, not their complete proofs. Exact versions, hashes and access limits are in the source archive. Bounded primary-source, public-record and duplicate checks on 22 October 2026 authenticated no earlier publication of this exact result. This is not an exhaustive novelty, exclusive-firstness, or current-openness certificate. The earlier project PR1348, head `4222445`, is an unpublished proof source, not an earlier published solution. Catalog 10000037/AMR-099-0037 is one result family. The source-question DOI is not a counterexample DOI.

The source archive supplies an exact standard-library checker for finite dual-boundary, series-connectivity, metric and rational-constant controls, with deliberately false mutations. These are supplementary diagnostics; the written argument supplies the unbounded paths, infinite events and uniform metric quantifiers. We do not assert almost-sure multiplicity, exact cluster count, critical nonuniqueness, or a simultaneous-in-p law.

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