Classical arrangement counterexamples to summed rough-M semicontinuity

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Abstract

The classical dual Hesse arrangement and the established deformation of an affine line arrangement to its pencil give a delta-constant multibranch family whose central rough invariant is 7, whereas the sum on every nearby fiber is 12. We record the elementary corollary with explicit coefficients, finite simultaneous normalization, and a common transverse ball. It gives a negative answer to the unrestricted summed reading of the rough-invariant question in Borodzik's 2009 Oberwolfach contribution. It does not address the later explicitly unibranched parametric conjecture. We credit the earlier construction and method and make no firstness or present-openness claim.

Question and provenance. For a reduced plane-curve singularity, let

$$\overline{M}(C,p) = K \cdot (K + D), \qquad D = C' + E_{\mathrm{red}},$$

on its minimal embedded resolution; K is the local canonical divisor supported on the exceptional divisor. For a fiber with several singularities we take the sum of these local values in a fixed representative. Borodzik's report [1, pp. 2446–2447] defines this rough invariant and asks about upper semicontinuity, including delta-constant deformations. The question itself does not write a sum or impose unibranchedness; its following fixed-ball discussion compares sums over nearby singularities. Our conclusion concerns precisely this unrestricted summed reading, without asserting an unstated intended scope.

The geometry is prior mathematics. De Jong–van Straten [3, Definition 4.2 and Example 6.6] use the exact nine-line, twelve-triple configuration in the delta-constant picture-deformation framework. The scaling of affine line constants to zero is explicit in Plamenevskaya–Starkston [4, Example 8.7]. Borodzik [2, p. 578] already uses contraction to an ordinary multiple point and its rough value $r - 2$. Thus the theorem below is an explicit classical corollary and formulation clarification; it asserts neither a new arrangement nor a new deformation method. The inspected sources did not establish earlier publication of this exact rough-invariant comparison; that bounded observation does not establish priority.

Theorem 1. *There is a reduced flat holomorphic family in the open unit ball in $\mathbb{C}^2$, over $|t| < 1/4$, with finite simultaneous normalization and smooth disjoint transverse boundary circles, such that the central fiber has an ordinary ninefold point and every nonzero fiber has exactly twelve ordinary triple points. All singularities lie inside the ball of radius $1/2$, and*

$$\delta(C_0, 0) = \sum_p \delta(C_t, p) = 36, \qquad \overline{M}(C_0, 0) = 7 < 12 = \sum_p \overline{M}(C_t, p) \quad (t \neq 0).$$

Every normalized fiber consists of nine disks. The construction restricts, after scaling, to arbitrarily small ball representatives.

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Proof. Let ω be a primitive cube root of unity. In $\mathbb{P}^2$ take

$$A_i : X - \omega^i Y = 0, \quad B_j : Y - \omega^j Z = 0, \quad C_k : Z - \omega^k X = 0 \qquad (i, j, k \bmod 3).$$

The three same-group intersections are the coordinate vertices. The other nine are

$$P_{ij} = [\omega^{i+j} : \omega^j : 1],$$

each incident to exactly A_i, B_j, C_{-i-j}. All twelve are ordinary triples, and they exhaust the $12\binom{3}{2} = \binom{9}{2}$ pairs. In particular there are no further intersections. This is the dual Hesse arrangement of [3].

The infinity $L = X + 2Y + 4Z = 0$ misses the vertices and every P_{ij}, since $|\omega^{i+j} + 2\omega^j + 4| \geq 1$. In the chart $u = X/L, v = Y/L$, all nine affine directions are distinct. Scaling $(u, v) = (x/t, y/t)$ gives the equation

$$F = (x^3 - y^3)\big(64y^3 - (t - x - 2y)^3\big)\big((t - x - 2y)^3 - 64x^3\big) = 0. \tag{1}$$

Its nine linear factors, indexed by $q \in \{1, \omega, \omega^2\}$, are

$$x - qy, \qquad qx + (4 + 2q)y - qt, \qquad -(1 + 4q)x - 2y + t.$$

For $t \neq 0$ its singular set is exactly the scaled twelve-point set; at zero the nine distinct directions meet only at the origin. Every fiber is reduced. The coefficient of x^9 is the constant -65, so the polynomial quotient is free of rank nine over $\mathbb{C}[y, t]$ and flat over $\mathbb{C}[t]$. Directly in the analytic local ring at parameter t_0, put $s = t - t_0$. If $sg = Fh$, reduction modulo s in the domain $\mathbb{C}\{x - x_0, y - y_0\}$ shows $h = sh_1$, since $F_{t_0} \neq 0$. Cancellation gives $g = Fh_1$. The quotient is torsion-free over the base discrete valuation ring, hence flat. This persists on open restriction.

Each linear factor has nonzero x-coefficient, so its component plane is a graph over (y, t). The disjoint union of the nine planes maps finitely to $F = 0$: on rings its algebra is the finite product of the nine cyclic component quotients. The component ideals intersect in their product, making this an injective extension with the same total fraction ring. The product algebra is normal, finite and integral over the reduced quotient, hence is its integral closure. At every fixed t, including zero, it restricts to the disjoint union of nine distinct smooth lines, which is the normalization of that fiber. This is finite simultaneous normalization, with normalized projection smooth. The total space $F = 0$ itself is nonnormal. Analytic restriction preserves this description.

Use the open unit ball for the holomorphic family, and its closed ball for boundary statements. The vertex chart coordinates are $(1, 0), (0, 1/2), (0, 0)$. At the other points, both coordinate numerators have modulus one and the common denominator has modulus at least one. Hence every singular point has norm at most $\sqrt{2}|t| < 1/2$. For any of the displayed lines $ax + by + ct = 0$, its distance to the origin is $d(t) = |ct|/(|a|^2 + |b|^2)^{1/2}$. The three groups have bounds respectively $0, |t|/\sqrt{5}, |t|/\sqrt{13}$, all strictly below one. A closest point and a fixed unit direction are

$$p(t) = -\frac{ct(\bar{a}, \bar{b})}{|a|^2 + |b|^2}, \qquad v = \frac{(-b, a)}{\sqrt{|a|^2 + |b|^2}}.$$

The sphere intersection is $p(t) + \sqrt{1 - d(t)^2}\, e^{i\theta} v$, a transverse smoothly varying circle. These nine circles are disjoint because all pair intersections are strictly inside the half-radius ball. Thus they give a link isotopy along any parameter path, and each normalized restricted line is a disk. The central cone is transverse to every sphere about zero. Homogeneity $F(\rho x, \rho y, \rho t) = \rho^9 F(x, y, t)$

gives the corresponding radius-ρ representative over $|t| < \rho/4$. No properness of the open-ball projection is needed; the normalization map is finite.

For an ordinary r-fold point, transverse smooth branches give $\delta_r = \binom{r}{2}$, by the additivity formula $\delta(fg) = \delta(f) + \delta(g) + I(f, g)$. For $r \geq 3$, one blowup has $E^2 = -1$, $E \cdot C' = r$, and $K = E$: adjunction gives $(K + E)E = -2$. Consequently

$$\overline{M}_r = E \cdot (C' + 2E) = r - 2.$$

A node already has normal crossings and rough value zero, in agreement with this formula at $r = 2$. Applying it at the ordinary ninefold point and all twelve triples proves both displayed equalities and the strict inequality. In particular delta constancy is an equality of total local invariants, with all singularities retained in the representative. □

Classical Fermat extension. For every integer $n \geq 3$, the classical arrangement

$$(X^n - Y^n)(Y^n - Z^n)(Z^n - X^n) = 0$$

has $3n$ lines, three ordinary n-fold vertices and n^2 other ordinary triples [5, p. 119]. Indeed the latter are $[\zeta^{i+j} : \zeta^j : 1]$; exactly one line of each group passes through each, and same-group pairs meet at the vertices. These exhaust all pairs. The same infinity misses every intersection. Scaling its affine factors gives, by the proof above, finite simultaneous normalization and a sufficiently small common transverse ball. Therefore

$$3\binom{n}{2} + 3n^2 = \binom{3n}{2}, \qquad \sum_p \overline{M}(C_t, p) - \overline{M}(C_0, 0) = 3(n-2) + n^2 - (3n-2) = n^2 - 4.$$

The excesses are unbounded. This conclusion follows from the all-n incidence argument, rather than a finite computational range. At $n = 2$ the vertices are nodes and the excess is zero.

Limits of the conclusion. The sums here run over all moving singularities; no counterexample to a marked-section assertion is claimed. The central curve has nine branches. Borodzik's published Definition 2.1 and Conjecture 3.8 [2, pp. 575,577] require a parametric deformation to have a *unibranched* central fiber as well as rational normalization. The present family fails that hypothesis and does not resolve that conjecture. We use precisely rough $K(K + D)$, without substituting a different multibranch convention or a fine invariant. For these ordinary points $K+D$ is relatively nef, so the negative part is zero; this does not identify fine-invariant conventions on other singularities.

Verification and review status. The companion source and verification archive contains the unchanged original 28-file packet, independent exact arithmetic controls, reviewed provenance records, and a verifier using externally anchored hashes and isolated nonoptimized replay. Computation checks incidence, factorization, invariant arithmetic and boundary estimates; it supplements the analytic proof and proves neither priority nor a universal assertion by sampling. Raw third-party source bodies are excluded. The original recovery record says "1 verified/5"; the earlier attempt count is unavailable. AI tools were used extensively in the research, writing and independent adversarial checking. Those checks are not independent human peer review. This preprint is unrefereed.

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