A vertex maximum principle for subcritical reinforcement

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Abstract

We give a self-contained alternative proof of positive-equilibrium uniqueness and almost-sure coordinatewise homogenization for subcritical reinforcement on countable graphs of bounded degree. Positive vertex firing rates are bounded above but may have infimum zero. The proof combines an exact compensated jump transform with a maximum principle for bounded positive complete trajectories in vertex coordinates. Initial edge counts are arbitrary finite positive integers. Qin has also proved the full homogenization theorem, with broader positive real initialization; that result is prominently credited here. Unrestricted uniqueness among nonnegative equilibria is false, and is kept separate from the corrected positive statement. No first-priority claim is made.

1 The question and the precise result

In the graph-based reinforcement model, each vertex has an independent Poisson clock. At a ring, one incident edge is chosen proportionally to its current weight raised to a power α, and its weight increases by one. Couzinié and Hirsch [1] proved homogenization on general bounded-degree graphs for $\alpha < 1/2$, slightly beyond $1/2$ on regular graphs, and throughout the subcritical range on $\mathbb{Z}$. Earlier heterogeneous-rate results and the all-subcritical conjecture appear in Couzinié's thesis [2]. The question for bounded positive heterogeneous firing rates is stated in the Oberwolfach report [3] and the official pre-publication version of the MATRIX chapter [4].

The report's literal equilibrium definition does not require positive coordinates. Unrestricted uniqueness is false. On $\mathbb{Z}$, with unit vertex rates and $0 < \alpha < 1$, put weight two on a perfect matching and zero on all other edges. Every incident denominator is 2^α. Each positive edge receives rate two, and each zero edge receives rate zero. Both phases and the constant-one array are equilibria. This is the same local matching mechanism as the older normalized alternating-cycle allocation in finite WARM [5, §3.6]; its validity for every positive exponent follows by direct substitution. Qin [6, Remark 1.3] also records the infinite matching example. We claim no novelty for this boundary construction.

The positive theorem below is an explicitly corrected homogenization formulation, not a proof of that false unrestricted uniqueness sentence. Qin [6, Theorem 1.1] proves this theorem and permits arbitrary finite positive real initial weights. His connected formulation extends componentwise to disconnected countable graphs. Our initialization class is smaller. The contribution here is a different proof: finite-volume vertex minimization, an exact jump compensator, and an approximate space-time maximum principle. The power/incidence reduction and complete-limit strategy are shared structural ideas, and are not asserted to originate here. Qin also reports independent Angel–Hermon–Yao work; its full manuscript and public date were not verified in our bounded literature audit.

Let $G = (V, E)$ be a countable undirected simple graph. Discard isolated vertices and assume

$$1 \le d(v) \le D < \infty, \qquad 0 < p_v \le P < \infty.$$

There is no positive lower bound on all p_v. For an edge array $w > 0$, set

$$S_v(w) = \sum_{e \ni v} w_e^\alpha.$$

The clocks have rates p_v, and choose $e \ni v$ with probability $N_e^\alpha / S_v(N)$. Initially $N_e(0) = n_e \in \mathbb{N}$, with $n_e \geq 1$ finite for each edge; no uniform bound on (n_e) is imposed. Write $X_e(t) = N_e(t)/t$.

Theorem 1. *For $0 \leq \alpha < 1$, there is exactly one equilibrium positive on every edge:*

$$x_{uv} = x_{uv}^\alpha \left(\frac{p_u}{S_u(x)} + \frac{p_v}{S_v(x)} \right). \tag{1}$$

Almost surely, $X_e(t) \longrightarrow x_e$ simultaneously for all $e \in E$. The convergence is coordinatewise.

The empty edge set gives the unique empty array and vacuous convergence. For the proof assume $E \neq \varnothing$ and $0 < \alpha < 1$, and put $c = 1 - \alpha$ and $\beta = \alpha/c > 0$. The endpoint $\alpha = 0$ is treated at the end. No uniform spatial convergence, stochastic rate, zero-initial-edge, critical-exponent or unbounded-degree extension is claimed.

2 Positive equilibria in vertex coordinates

For a positive equilibrium define $z_v = p_v / S_v(x)$. Dividing (1) by x_{uv}^α gives the bijective correspondence

$$x_{uv} = (z_u + z_v)^{1/c}, \qquad z_v T_v(z) = p_v, \qquad T_v(z) = \sum_{w \sim v} (z_v + z_w)^\beta. \tag{2}$$

Every positive vertex solution obeys

$$z_v \leq (p_v / d(v))^c \leq B := P^c, \tag{3}$$

since $T_v(z) \geq d(v) z_v^\beta$. Thus boundedness is a deduction for every positive equilibrium, rather than an extra hypothesis.

Proposition 2. *The vertex equations in (2) have exactly one positive solution.*

Proof. For a finite vertex set F, set $z_w = 0$ outside F and minimize

$$\Phi_F(z) = \sum_{\{u,v\} \in E : \{u,v\} \cap F \neq \varnothing} \frac{(z_u + z_v)^{\beta+1}}{\beta + 1} - \sum_{v \in F} p_v \log z_v \qquad (z_v > 0, v \in F).$$

Only finitely many edges occur. The power sum dominates a positive multiple of $\sum_{v \in F} z_v^{\beta+1}$ and hence the logarithms at infinity. On bounded sublevel sets, approaching a coordinate hyperplane costs infinity. An interior minimum exists. The logarithmic terms are strictly convex and the power terms are convex, so the minimum is unique. Its equations are precisely $z_v T_v(z) = p_v$ for $v \in F$.

The upper estimate (3) holds, as does

$$z_v \geq \frac{p_v}{d(v)(2B)^\beta} > 0 \qquad (v \in F).$$

Exhaust V by finite sets and take a diagonal subsequence of their minimizers. Each fixed coordinate eventually has this positive lower bound and the common upper bound B. Every vertex equation is a finite sum, so it passes to the coordinatewise limit. This proves positive infinite-volume existence.

For uniqueness let z, y be positive solutions and suppose $M = \sup_v |z_v - y_v| > 0$. Interchange them if necessary and choose, for arbitrarily small $\varepsilon > 0$, a vertex with $z_v - y_v > M - \varepsilon$.

Each adjacent pair sum for z is at least the corresponding sum for y minus ε. Let ω be a modulus of continuity of r^β on $[0, 2B]$. Then $T_v(z) - T_v(y) \geq -D\omega(\varepsilon)$, while $z_v \geq M - \varepsilon$ and $T_v(z) \geq (M - \varepsilon)^\beta$. Consequently

$$0 = (z_v - y_v)T_v(z) + y_v[T_v(z) - T_v(y)] \geq (M - \varepsilon)^{\beta+1} - BD\omega(\varepsilon),$$

which is positive for small ε. No maximum attainment or uniform positive lower bound is used. □

3 An exact stochastic transform

The graphical construction is local. A fixed query up to time T has causal predecessors along vertex-clock paths with at most $D + 1$ choices at each step and strictly decreasing times. The expected number of length-m ordered paths is bounded by a fixed local factor times $((D + 1)PT)^m/m!$, tending to zero. The predecessor tree is finitely branching; infinite ancestry would give an infinite path. Thus local queries have finite ancestry almost surely. Marks resolve those finite histories. Countably many edge queries and integer horizons give a consistent process, even when the total firing rate is infinite. This clock construction does not require bounded initial counts.

For $e = \{u, v\}$, endpoint-clock strong laws imply

$$\limsup_{t\to\infty} X_e(t) \leq p_u + p_v \leq 2P. \tag{4}$$

Use predictable left limits to define

$$q_e(t) = \frac{p_u}{S_u(N(t-))} + \frac{p_v}{S_v(N(t-))}, \qquad Q_e(t) = \int_0^t q_e(s)\, ds.$$

The edge count has intensity $N_e(t-)^\alpha q_e(t) \leq p_u + p_v$. For positive integers let

$$H(n) = \sum_{j=1}^{n-1} j^{-\alpha} = \frac{n^c}{c} + O(1).$$

Its exact jump is $n^{-\alpha}$. Therefore

$$H(N_e(t)) - H(n_e) = Q_e(t) + M_e(t), \qquad \langle M_e\rangle(t) = \int_0^t N_e(s-)^{-\alpha}\, dQ_e(s) \leq Q_e(t), \tag{5}$$

where $M_e(0) = 0$. Since $N_e \geq 1$ and $q_e \leq 2P$, all finite-horizon martingales here are square integrable.

Whenever $Q_e \to \infty$, (5) gives $M_e/Q_e \to 0$ almost surely. Indeed, with $A = 1 + Q_e$, the martingale $L(t) = \int_0^t A(s)^{-1}\, dM_e(s)$ has bracket at most $\int_0^\infty (1 + Q_e)^{-2}\, dQ_e \leq 1$. It is bounded in L^2 and converges almost surely. Integration by parts gives

$$\frac{M_e(t)}{A(t)} = L(t) - \frac{1}{A(t)} \int_0^t L(s)\, dA(s).$$

On $A(t) \to \infty$, weighted Cesaro convergence makes the two terms have the same limit. This argument does not assume independence of Q_e and M_e.

Fix $\varepsilon > 0$. By (4), eventually all finitely many edges incident to u or v have counts at most $(2P + \varepsilon)t$. Thus

$$q_e(t) \geq k_{e,\varepsilon} t^{-\alpha}, \qquad k_{e,\varepsilon} = \frac{p_u/d(u) + p_v/d(v)}{(2P + \varepsilon)^\alpha} > 0.$$

This first proves $Q_e \to \infty$ and $\liminf Q_e(t)/t^c \geq k_{e,\varepsilon}/c$. The transform and its normalized martingale law then imply $N_e(t)^c/(cQ_e(t)) \to 1$, and hence

$$\liminf_{t\to\infty} X_e(t) \geq m_e := \left(\frac{p_u/d(u) + p_v/d(v)}{(2P)^\alpha}\right)^{1/c} > 0. \tag{6}$$

The upper bound also gives $H(N_e(t)) = O(t^c)$, so $Q_e(t) = O(t^c)$ and $M_e(t)/t^c \to 0$. All statements hold on one probability-one event by countability. The constants and random starting times need not be uniform in space. The fixed initial term $H(n_e)$ vanishes after scaling.

4 Complete limiting trajectories

Define nonnegative vertex primitives

$$A_v(t) = ct^{-c}\int_0^t \frac{p_v}{S_v(N(s))}\,ds.$$

Left limits are immaterial in this Lebesgue integral. Splitting Q_e in (5) gives

$$X_{uv}(t)^c = A_u(t) + A_v(t) + r_{uv}(t), \qquad r_{uv}(t) \longrightarrow 0. \tag{7}$$

Nonnegativity and (4) imply $\limsup A_v(t) \leq K := (2P)^c$. Integrating the local incident-count upper bound similarly gives

$$\liminf_{t\to\infty} A_v(t) \geq \ell_v := \frac{p_v}{d(v)(2P)^\alpha} > 0.$$

In logarithmic time $a_v(s) = A_v(e^s)$, differentiation holds almost everywhere:

$$a_v'(s) = c\left[\frac{p_v}{S_v(X(e^s))} - a_v(s)\right]. \tag{8}$$

The finitely many incident-edge lower bounds (6) bound this derivative for each fixed vertex at all sufficiently large times. Thus the translated coordinate functions are eventually uniformly Lipschitz on each compact interval, with a bound allowed to depend on the coordinate.

For any $s_n \to \infty$, Arzela–Ascoli and a diagonal subsequence over the countably many vertices and intervals $[-j, j]$ give locally uniform coordinate limits $a_v(s_n + s) \to b_v(s)$, defined for every $s \in \mathbb{R}$, with

$$\ell_v \leq b_v(s) \leq K.$$

For each edge the error in (7) is uniformly small on a shifted compact logarithmic interval, because its minimum physical time tends to infinity. Since $X_{uv}^\alpha = (X_{uv}^c)^\beta$, finite incident sums and positive limiting coordinates justify passing through the denominators in the integral form of (8). Every limit is a classical complete trajectory of

$$b_v' = c\,[R_v(b) - b_v], \qquad R_v(b) = \frac{p_v}{T_v(b)}. \tag{9}$$

This is product-topology compactness. No spatially uniform random time, supremum-norm compactness or infinite-dimensional stochastic approximation theorem has been assumed.

5 The vertex maximum principle and convergence

Lemma 3. *Every positive complete classical solution of (9) with a common finite upper bound is the equilibrium z of Proposition 2. No common positive lower bound is required.*

Proof. Choose a common upper bound $K_0 \geq B$ for b and z, and suppose $M = \sup_{v,s} |b_v(s) - z_v| > 0$. Set $d_v = b_v - z_v$. Let ω be a modulus for r^β on $[0, 2K_0]$. Choose $0 < \varepsilon < M/4$ so that

$$\frac{BD\omega(\varepsilon)}{(M/2)^\beta} < M/4.$$

If $d_v \geq M - \varepsilon$, every neighbor has $d_w \geq -M$, whence $T_v(b) \geq T_v(z) - D\omega(\varepsilon)$. Also $b_v > M/2$, so $T_v(b) \geq (M/2)^\beta$. Using $p_v = z_v T_v(z)$ gives

$$R_v(b) - z_v = \frac{z_v[T_v(z) - T_v(b)]}{T_v(b)} < M/4, \qquad d_v' \leq -cM/2.$$

If $d_v \leq -M + \varepsilon$, every neighbor has $d_w \leq M$, whence $T_v(b) \leq T_v(z) + D\omega(\varepsilon)$. Here $z_v > M/2$, so $T_v(z) \geq (M/2)^\beta$. Reciprocal monotonicity gives

$$R_v(b) - z_v \geq -\frac{z_v D\omega(\varepsilon)}{T_v(z) + D\omega(\varepsilon)} > -M/4, \qquad d_v' \geq cM/2.$$

In particular, the lower-strip estimate needs no uniform lower bound on the current denominator $T_v(b)$.

Some fixed coordinate at some time lies strictly in one extreme strip. In the upper case, take the connected component of $\{s : d_v(s) > M - \varepsilon\}$ containing that time. A finite left endpoint would have value $M - \varepsilon$; integrating the negative derivative from it contradicts entry into the strip. The component therefore extends to minus infinity. Integrating backward gives arbitrarily large d_v, contradicting $d_v \leq M$. The lower case is identical with the sign reversed. Thus $M = 0$. Uniform continuity suffices even for $0 < \beta < 1$; neither an attained maximum nor differentiation of an infinite supremum is used. □

Every limiting trajectory from the previous section satisfies the lemma and equals z. Hence $a_v(s) \to z_v$ for each vertex: a sequence separated from z_v would have a limiting trajectory separated at time zero, a contradiction. Equation (7) then gives $X_{uv}(t) \to (z_u + z_v)^{1/c}$. Proposition 2 identifies this as the unique positive equilibrium. For $\alpha = 0$, independent marking and Poisson thinning directly give

$$X_{uv}(t) \longrightarrow p_u/d(u) + p_v/d(v),$$

which is also the unique positive equilibrium. This completes the proof of Theorem 1.

Chronology, verification and disclosure

The candidate argument is recorded in repository PR 764 [7], whose current GitHub server creation and update timestamps are 5 October 2026; Qin's arXiv submission is dated 8 October 2026. These observations do not certify when every current proof byte first became publicly available, or historical priority over all work. The available literature now contains the full positive theorem. Our note presents an alternative proof and makes no first, sole, new-theorem or currently-open claim. The AHY manuscript reported by Qin remains a bibliographic limitation, not a purportedly inspected proof. The final publisher MATRIX chapter body was not obtained; references to its question's wording concern the official hosted pre-publication version.

The companion archive supplies the source, frozen candidate, dated mathematical and priority audits, bibliographic receipts, and reproducible finite and integrity controls. Computation does not certify the infinite stochastic proof or exhaustive priority. AI tools were used extensively in development, writing, reproduction, verification and adversarial review. This is an unrefereed research note: it has not undergone external human peer review or formal proof-assistant certification. The note and companion materials are released under CC BY 4.0.

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