An explicit SU(5) lens-space counterexample to the printed Guadagnini–Pilo conjecture

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Abstract

For the ordinary full SU(5) Reshetikhin–Turaev invariant at WZW level $k = 5$, or shifted level $r = 10$, the lens spaces $L(5, 1)$ and $L(5, 2)$ have fundamental group $\mathbb{Z}/5$ but unequal nonzero magnitudes. With $\widehat{\tau}(S^3) = 1$, their squared magnitudes are $3475+1550\sqrt{5}$ and $4025+1800\sqrt{5}$. A short exact root-lattice reduction proves the calculation, conditional on the established modular-category and surgery formulas; finite cyclotomic programs provide reproducible checks. This contradicts Conjecture 7.5 as printed in Ohtsuki's collection. Historical priority is unresolved: Takahito Kuriya's directly relevant preprint could not be obtained, and no firstness or new historical resolution is claimed. This is an unrefereed research note with extensive AI assistance and no conventional human peer review.

1 Claim and conventions

Ohtsuki's edited collection [1] prints Conjecture 7.5 on p. 474, attributed to Guadagnini and Pilo [2]: the absolute value of a nonzero quantum G invariant of a closed oriented 3-manifold is determined by its fundamental group. The volume is nominally dated 2002 and records publication on June 1, 2004. We refer to this printed claim, without asserting its present literature status. Its scope includes compact simple simply connected $G = \mathrm{SU}(5)$; it is not restricted to the SU(2) lens-space theorem in [2].

Theorem 1. *In the full ordinary SU(5) theory at WZW level $k = 5$, shifted level $r = k + h^\vee = 10$, let $\widehat{\tau}(M) = \tau(M)/\tau(S^3)$. Then*

$$|\widehat{\tau}(L(5,1))|^2 = 3475 + 1550\sqrt{5}, \tag{1}$$

$$|\widehat{\tau}(L(5,2))|^2 = 4025 + 1800\sqrt{5}. \tag{2}$$

Both manifolds have fundamental group $\mathbb{Z}/5$. Both values are positive, and their difference is $550 + 250\sqrt{5} > 0$. Thus the printed nonzero-magnitude claim fails.

We use the conventions of Hansen–Takata [3], specifically version 2 of their available preprint. Its final journal version has not been compared here. The invariant τ there has $\tau(S^3) = S_{00} > 0$. Their $L(p, q)$ is unknot surgery with coefficient $-p/q$. A convention using $+p/q$ reverses orientation and conjugates the ordinary unitary invariant, leaving the magnitudes unchanged. The quantum-group root is $e^{\pi i/10}$; the renormalized coupling in Guadagnini–Pilo notation corresponds to shifted $r = 10$, not WZW $k = 5$.

The full category has all dominant labels (a_1, a_2, a_3, a_4) with $a_i \geq 0$ and $\sum_i a_i \leq 5$, hence $\binom{9}{4} = 126$ labels. No root-lattice restriction on those labels, projective quotient, bundle sector, spin choice, inserted link, or retained manifold framing is used. The root lattice below is a computational quotient in a formula for this full category. The existence and identification of the category, and the general RT surgery theorem, are standard imported mathematical inputs from [3]; we do not rebuild those foundations.

2 Exact lattice reduction

Realize the A_4 root system in $H = \{x \in \mathbb{R}^5 : \sum_i x_i = 0\}$ with root lattice $Y = H \cap \mathbb{Z}^5$, weight lattice $X = Y^*$, and $\rho = (2, 1, 0, -1, -2)$. Roots have squared length 2, $|\rho|^2 = 10$, rank is 4, and there are 10 positive roots. The simple-root Gram determinant is 5, so $\mathrm{vol}(Y) = \sqrt{5}$ and $\mathrm{vol}(X) = 1/\sqrt{5}$.

Write $\mathcal{S}(q/p) = 12s(q, p)$ for the Dedekind symbol. For $p > 0$, $s(q, p) = \sum_{n=1}^{p-1}((n/p))((qn/p))$, with $((x)) = x - \lfloor x \rfloor - 1/2$ off integers and zero at integers. Hansen–Takata Theorem 5.1 specializes to

$$\tau(L(p, q)) = \frac{i^{10} e^{\pi i \mathcal{S}(q/p)}}{(10p)^2 \sqrt{5}} \sum_{w \in S_5} \det(w) e^{-2\pi i \langle \rho, w\rho \rangle/(10p)} \sum_{\nu \in Y/pY} e^{\pi i (10q/p)|\nu|^2} e^{(2\pi i/p)\langle \nu, q\rho - w\rho \rangle}. \tag{3}$$

Here $\gcd(p, q) = 1$ and $p \neq 0$. This theorem has no coprimality requirement between p and r; the coprime simplification in their Proposition 5.2 is inapplicable when $p = 5, r = 10$ and is not used.

Set $p = 5$. Representatives of $Y/5Y$ are

$$\nu(a) = (a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3, -a_4), \quad 0 \leq a_i < 5.$$

The simple roots are a free integral basis, so this gives all $5^4 = 625$ classes. The quadratic phase in (3) is $e^{2\pi i q|\nu|^2} = 1$. For an integral sum-zero vector v, character orthogonality gives

$$\sum_{\nu \in Y/5Y} e^{2\pi i \langle \nu, v \rangle/5} = \prod_{j=1}^{4} \sum_{a=0}^{4} e^{2\pi i a(v_j - v_{j+1})/5} = \begin{cases} 625, & v_1 \equiv \cdots \equiv v_5 \pmod 5, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The criterion is $v \in 5X$, not $v \in 5Y$. Since ρ exhausts the five residues, precisely five permutations survive for each nonzero $q \pmod 5$.

Put $z = e^{2\pi i/50}$, $t = z^5 = e^{\pi i/5}$, and

$$B_q = \sum_{w:\ q\rho - w\rho \in 5X} \det(w) z^{-\langle \rho, w\rho \rangle}.$$

Table 1 lists every survivor for $q = 1, 2$. Signs are permutation parities; dot products are elementary integer calculations. Thus

$$B_1 = t^{-2} + 2 + 2t, \qquad B_2 = -(1 + 2t^{-1} + 2t). \tag{5}$$

In the positive real embedding $t + t^{-1} = (1 + \sqrt{5})/2$. Consequently

$$|B_1|^2 = 11 + 2\sqrt{5}, \qquad |B_2|^2 = 9 + 4\sqrt{5}. \tag{6}$$

For the first identity, expand the norm as $9 + 4(t + t^{-1}) + 2(t^2 + t^{-2}) + 2(t^3 + t^{-3})$ and use $t^2 + t^{-2} = (\sqrt{5} - 1)/2$ and $t^3 + t^{-3} = -(t^2 + t^{-2})$. For the second, $B_2 = -(2 + \sqrt{5})$.

q	$w\rho$	$\det(w)$	$\langle\rho, w\rho\rangle$
1	$(2,1,0,-1,-2)$	$+1$	10
1	$(1,0,-1,-2,2)$	$+1$	0
1	$(0,-1,-2,2,1)$	$+1$	-5
1	$(-1,-2,2,1,0)$	$+1$	-5
1	$(-2,2,1,0,-1)$	$+1$	0
2	$(2,0,-2,1,-1)$	-1	5
2	$(1,-1,2,0,-2)$	-1	5
2	$(0,-2,1,-1,2)$	-1	-5
2	$(-1,2,0,-2,1)$	-1	0
2	$(-2,1,-1,2,0)$	-1	-5

Table 1: The five Weyl survivors for each target lens space.

3 Normalization and topology

At $p = 1$ the quotient has one class and the Dedekind symbol is zero. Let

$$A = \sum_{w \in S_5} \det(w) e^{-2\pi i \langle \rho, w\rho \rangle / 10}.$$

The Weyl denominator identity gives $A = -P$, where

$$P = \prod_{j=1}^{4} (2\sin(\pi j/10))^{5-j} = 5\sqrt{5} - 10 > 0.$$

Indeed $(2\sin 18^\circ)(2\sin 54^\circ) = 1$, $(2\sin 36^\circ)(2\sin 72^\circ) = \sqrt{5}$, and $(2\sin 36^\circ)^2 = (5-\sqrt{5})/2$. Therefore

$$A = 10 - 5\sqrt{5}, \quad |A|^2 = 225 - 100\sqrt{5}, \quad s = S_{00} = \tau(S^3) = \frac{-A}{100\sqrt{5}} > 0. \tag{7}$$

Canceling the prefactors of (3) at $p = 5$ and $p = 1$ gives

$$\widehat{\tau}(L(5,q)) = 25 e^{\pi i \mathcal{S}(q/5)} B_q / A, \qquad |\widehat{\tau}(L(5,q))|^2 = \frac{625 |B_q|^2}{225 - 100\sqrt{5}}. \tag{8}$$

Since $(225 - 100\sqrt{5})(225 + 100\sqrt{5}) = 625$, equations (6) and (8) prove (1)–(2). The denominator is positive, for $225^2 > 100^2 \cdot 5$.

For comparison with vacuum matrix words, the positive rank is $D = s^{-1} = 100 + 40\sqrt{5}$, and $s^2 = (9 - 4\sqrt{5})/2000$. The ordinary normalized modular matrix is

$$S_{\lambda\mu} = -\frac{1}{100\sqrt{5}} \sum_{w \in S_5} \det(w) e^{-2\pi i \langle w(\lambda+\rho), \mu+\rho \rangle / 10}, \quad \theta_\lambda = e^{\pi i (|\lambda+\rho|^2 - |\rho|^2)/10}.$$

The categorical Hopf matrix is DS, not S. The two positive surgery chains are $5 = [5]$ and $5/2 = [3,2]$, giving words $S\theta^5 S$ and $S\theta^3 S\theta^2 S$. For either word F, the S^3-normalized magnitude is $|F_{00}|/s$, with exactly one division by s. Hansen–Takata Corollary 4.2 and Eqs. (26)–(30), (35) supply this full-category normalization. Here central charge is $5 \cdot 24/10 = 12$, so the linear T is $-\theta$ and the anomaly $\Delta/D = -1$. These and the signature corrections have modulus one.

Replacing q by $-q$ conjugates (3): use the longest Weyl element, which sends ρ to $-\rho$ and has sign $+1$, and $s(-q,5) = -s(q,5)$. Thus $q = 4$ gives the same magnitude as $q = 1$, and $q = 3$ the same as $q = 2$. Inverse-denominator conventions also exchange 2 and 3. Neither convention removes the discrepancy. A fixed nonzero theory-wide rescaling cannot make unequal magnitudes equal.

Finally, the unknot exterior has fundamental group generated by its meridian μ, and its preferred longitude is nullhomotopic. Filling at $-5/q$ kills μ^5, so van Kampen gives $\pi_1(L(5,q)) = \langle \mu \mid \mu^5 = 1 \rangle \cong \mathbb{Z}/5$. These are connected closed oriented manifolds. No homotopy-equivalence, minimality, or all-rank classification assertion is needed.

4 Exact certificates and limits

The accompanying archive contains two full root-lattice checks using Python's standard library and an independent full-weight modular-word calculation using NumPy object arrays of arbitrary-precision integers. The latter enumerates all 126 labels, 120 Weyl permutations and 8,001 symmetric matrix entries in $\mathbb{Z}[\zeta_{100}]$, reduced modulo $\Phi_{100} = x^{40} - x^{30} + x^{20} - x^{10} + 1$. The lattice checks use Φ_{10} and Φ_{50} and enumerate all 625 quotient representatives; the fresh Φ_{50} route also checks orientation, inverse-denominator, periodicity and S^3 boundaries.

Validation uses explicit exceptions that remain active under Python `-O`. A manifest-bound runner records fresh child-process exit codes, streams, source hashes and runtime information in both modes; deliberately false arithmetic targets must fail in both modes. The historical output field `exact_assertions=2005` in one checker counts its explicit finite checks; it is not a quality measure or 2,005 independent mathematical results. Repetition in two modes does not double the distinct checks. Floating numbers in the fresh lattice output are labeled diagnostics and decide no exact identity or sign. The finite computations check this specialization; they do not prove the general modular-category theorem or establish historical priority.

5 Priority, assistance, and review

Takahito Kuriya's *The LMO invariant and the Guadagnini–Pilo conjecture for lens spaces* is listed as a preprint by his Kyushu institutional record [4]. An institutional report records an exact-title talk on February 21, 2003 [5]. Its full text and complete conclusions and hypotheses could not be obtained. We cannot establish that it does not already contain or circumscribe this result. Historical priority therefore remains unresolved. Titles, aggregator abstracts, later perturbative results, and bounded searches cannot replace that comparison. No first counterexample, exhaustive novelty clearance, or new historical resolution is claimed.

The separate priority supplement documents the read boundaries. In particular, ordinary $\mathrm{SU}(N)$ WZNW is explicitly spin-independent in the later Kubo–Yokoyama comparison [6]; their proposed background phases are a separate issue. Those $q = 1$ tables do not establish an earlier instance of this target pair. We make no blanket dismissal of $\mathrm{SU}(N)$ calculations as spin invariants.

AI tools were used extensively in solving, drafting, literature comparison, exact reproduction, and adversarial verification. Three separate AI validation families checked the arithmetic, normalization, and root-lattice/topological argument. This note has not undergone conventional human peer review and is unrefereed. The author takes responsibility for the stated claims and qualifications. The original author effort was two of five substantive approaches; preparation and validation added no central proof-search turn.

References

[1] T. Ohtsuki (ed.), *Problems on invariants of knots and 3-manifolds*, Geometry & Topology Monographs **4** (nominal 2002; published June 1, 2004), 377–572; Conjecture 7.5, p. 474. https://msp.org/gtm/2002/04/gtm-2002-04-024s.pdf.

[2] E. Guadagnini and L. Pilo, *Three-manifold invariants and their relation with the fundamental group*, Commun. Math. Phys. **192** (1998), 47–65. DOI: 10.1007/s002200050290. Available preprint: arXiv:hep-th/9612090v1.

[3] S. K. Hansen and T. Takata, *Reshetikhin–Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras, and their asymptotic expansions*, J. Knot Theory Ramifications **13**(5) (2004), 617–668. DOI: 10.1142/S0218216504003342. Formula source used here: arXiv:math/0209403v2, Eq. (12), Sections 4–5, Theorem 5.1 (preprint p. 39).

[4] T. Kuriya, *The LMO invariant and the Guadagnini–Pilo conjecture for lens spaces*, preprint, associated with a 2003 talk; full text unavailable to this work. Institutional listing: https://www2.math.kyushu-u.ac.jp/coe/staff/staff_sub/kuriya.html.

[5] Kyushu University 21st Century COE program, institutional report, Kuriya activities, printed p. 121, recording the February 21, 2003 talk. https://www2.math.kyushu-u.ac.jp/coe/report/pdf3/report-6.pdf.

[6] N. Kubo and S. Yokoyama, *Topological phase, spin Chern–Simons theory and level rank duality on lens space*, JHEP **04** (2022), 074. DOI: 10.1007/JHEP04(2022)074; arXiv:2108.09300v2.