Strong approximate innerness of finite abelian diagonal actions on Cuntz–Krieger algebras

Alec Kriebel
Independent researcher
ORCID: 0009-0001-9320-500X

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Abstract

We give a complete affirmative answer to the historical Question 1 in Barlak's Oberwolfach report on diagonal quasi-free automorphisms: every finite abelian diagonal quasi-free action on a simple purely infinite Cuntz–Krieger algebra associated with a finite essential 0–1 matrix is strongly approximately inner. The proof is an explicit deduction from Gabe–Szabó's equivariant uniqueness theorem, together with the established diagonal-coboundary mechanism and elementary finite-group averaging. It covers periodic matrices, nonouter actions and nonfaithful actions. This note is an exposition and verification of consequences of prior theorems; it makes no claim of a first solution or a new mechanism.

AI-use and verification disclosure. Extensive AI assistance was used in mathematical reasoning, proof drafting, source auditing, diagnostic computations and manuscript review. This manuscript is unrefereed; no external human peer review or formal proof-assistant certification is claimed. Finite computations in the accompanying verification material are diagnostics, not proofs of the operator-algebra assertions.

1 Statement and prior results

An action $\alpha : G \curvearrowright A$ on a unital C^*-algebra is *strongly approximately inner* if, for every $g \in G$, there are $u_{g,n} \in \mathcal{U}(A^\alpha)$ such that

$$\lim_n \|u_{g,n} a u_{g,n}^* - \alpha_g(a)\| = 0 \qquad (a \in A).$$

Here A^α is fixed by the *entire* action. The sequences for different g need not satisfy any approximate representation relation. This is the property in Izumi's definition and in [2, Definition 2.5]; it is weaker than approximate representability.

Let M be a finite 0–1 matrix with no zero rows or columns. Write $\mathcal{O}_M$ for its Cuntz–Krieger algebra, with canonical generators s_i and mutually orthogonal range projections $p_i = s_i s_i^*$ summing to 1, where $s_i^* s_i = \sum_j M(i,j) p_j$. We use the original workshop definition of diagonal quasi-free: the action preserves $\operatorname{span}\{s_i\}$ and fixes every p_i.

Theorem 1. *If $\mathcal{O}_M$ is simple and purely infinite, every action of a finite abelian group on $\mathcal{O}_M$ by these diagonal quasi-free automorphisms is strongly approximately inner.*

This answers the full question in [1, p. 2621, Question 1], without the outerness and aperiodicity assumptions of the adjacent positive result. The report has bibliographic year 2017 and was published online on 5 July 2018. Izumi had already established the standard finite Cuntz algebra case without outerness; the precise inspected source is his 2002 preprint, Proposition 5.6(2) [8]. Barlak–Szabó proved the outer aperiodic Cuntz–Krieger and graph cases [2, Theorems 1–2]; these are Corollaries 2.7–2.8 in their arXiv v3.

Our shorter proof uses the general uniqueness theorem of Gabe–Szabó, first available in 2022 [6]. It does not assert that they printed Theorem 1. The diagonal-inner argument below is the finite-graph form of Matsumoto's earlier coboundary and periodic-point mechanism [10, Lemma 5.14 and Proposition 6.4]. An alternative explicit return-corner proof from the Barlak–Szabó theorem is retained in the verification material. Its graph corner is a special case of Crisp–Gow [5, Theorem 3.1 in v1]. These are prior mechanisms. The present source audit does not establish priority for an explicit older statement of the combined full theorem. In particular, the full Zacharias chapter and Izumi's final 2004 typeset body were not available in that audit; negative search results give no priority certificate.

2 Two finite-group facts

We first record the exact fixed-algebra correction that will be used twice. If x_n are unitaries and $\max_{h\in G}\|\gamma_h(x_n)-x_n\|\to 0$ for a finite-group action γ, then

$$b_n=|G|^{-1}\sum_{h\in G}\gamma_h(x_n)\in A^\gamma,\qquad \|b_n-x_n\|\longrightarrow 0.$$

Thus b_n is eventually invertible and its polar unitary $b_n(b_n^*b_n)^{-1/2}$ belongs to A^γ and is norm-close to x_n. Replacing x_n by this unitary preserves any point-norm implementing limit.

Lemma 2. *Let G be finite abelian and $\beta: G\curvearrowright A$ strongly approximately inner. If $d: G\to\mathcal{U}(A)$ is an honest β-cocycle, so $d_{gh}=d_g\beta_g(d_h)$ and $d_1=1$, then $\alpha_g=\mathrm{Ad}(d_g)\beta_g$ is strongly approximately inner.*

Proof. Fix g and choose $v_n\in\mathcal{U}(A^\beta)$ implementing β_g in the point-norm limit. Put $t_n=d_gv_n$. These unitaries implement α_g asymptotically. For $h\in G$, abelianness and the cocycle law give

$$\alpha_h(t_n)=d_h\beta_h(d_g)v_nd_h^*=d_g\beta_g(d_h)v_nd_h^*.$$

Consequently

$$\|\alpha_h(t_n)-t_n\|=\|\beta_g(d_h)-v_nd_hv_n^*\|\longrightarrow 0.$$

There are only finitely many h. Averaging over α and taking the polar unitary as above produces unitaries in A^α with the same implementing limit. □

The hypothesis is an actual cocycle law, not a scalar-projective law for arbitrary pointwise inner implementers. It imposes no group law on the final approximating sequences. The converse follows by applying the lemma to the honest α-cocycle $g\mapsto d_g^*$. The general framework of [4, Remark 3.2 and Theorem 3.7] supplies prior duality results for precisely this weaker strong approximate innerness property; the displayed elementary proof makes the transfer needed here explicit.

Lemma 3. *Let G be finite abelian and let $\gamma: G\curvearrowright A$ be a pointwise outer action on a unital Kirchberg algebra. Suppose every γ_g is homotopic to id_A through unital G-equivariant homomorphisms. Then γ is strongly approximately inner.*

Proof. Apply Gabe–Szabó's unital equivariant uniqueness theorem [6, Theorem 5.8] to the cocycle embeddings $(\mathrm{id}_A,1)$ and $(\gamma_g,1)$, separately for each g. Here G is exact; A is separable, unital and exact; the finite-group action is amenable; and pointwise outerness implies isometric shift absorption by [6, Proposition 3.14 in v1]. The proposed homotopy is equivariant, so the two embeddings have equal KK^G classes. Theorem 5.8 and Definition 1.15 of that source give unitaries $u_t\in\mathcal{U}(A)$ with

$$\mathrm{Ad}(u_t)(a)\longrightarrow\gamma_g(a),\qquad \max_{h\in G}\|\gamma_h(u_t)-u_t\|\longrightarrow 0.$$

The second conclusion is the cocycle-defect limit for the two trivial cocycles. The target is unital, so these are unitaries in A itself. Average and take polar unitaries, then sample times tending to infinity. This gives the required sequence in A^γ. □

For clarity, the cited v1 locators are Definition 1.15, p. 13; Proposition 3.14, p. 29; and Theorem 5.8, pp. 45–46. In v3 the proposition is numbered 3.15 and Theorem 5.8 is on p. 46. We use these inspected preprint statements, with their version-specific numbers, rather than guessing final article numbers. The unital theorem has no UCT hypothesis, additional stable-target hypothesis, or requirement that the implementing path start at 1.

3 Edge phases and the inner kernel

Let E be a finite strongly connected directed graph with at least one edge which is not a directed cycle. Use graph generators p_v, S_e with

$$S_e^* S_e = p_{r(e)}, \qquad p_v = \sum_{s(e)=v} S_e S_e^*.$$

Distinct edges have orthogonal range projections. For a path μ, use S_μ for its product, and include vertices as paths of length zero. An edge-diagonal action has the form

$$\alpha_g(p_v) = p_v, \qquad \alpha_g(S_e) = \chi_e(g) S_e,$$

where $\chi_e : G \to \mathbb{T}$ is a character. Set $\chi_\mu = \prod_{e \text{ in } \mu} \chi_e$, with vertex character 1.

Lemma 4. *If $\sigma(S_e) = z_e S_e$ and $\sigma(p_v) = p_v$, then σ is inner if and only if there are $t_v \in \mathbb{T}$ with $z_e = t_{s(e)} \overline{t_{r(e)}}$. In that case $\sigma = \mathrm{Ad}(\sum_v t_v p_v)$. The t_v are unique after normalizing one of them to* 1.

Proof. Every cycle has an exit: otherwise strong connectivity would make that exitless cycle the whole graph. The path diagonal $D_E = \overline{\mathrm{span}}\{S_\mu S_\mu^*\}$ is therefore a MASA [7, Theorem 5.2]; see also the convention-compatible statement in [3, §2.1]. Since σ fixes D_E pointwise, an inner implementer h must lie in $\mathcal{U}(D_E)$.

Identify D_E with $C(E^\infty)$, with path-range projections the cylinder functions. The relation $hS_\mu = z_\mu S_\mu h$ gives $h(\mu x) = z_\mu h(x)$ for every infinite tail x starting at $r(\mu)$. For a closed path μ choose $x = \mu\mu\cdots$. Then $\mu x = x$ and $|h(x)| = 1$, so $z_\mu = 1$.

Fix a base vertex b and directed paths $\theta_v : b \to v$, with θ_b empty. Put $t_v = z_{\theta_v}^{-1}$. If $e : v \to w$, append any path $\eta : w \to b$. Both $\theta_v e \eta$ and $\theta_w \eta$ are closed, giving $z_{\theta_v} z_e = z_{\theta_w}$ and hence the claimed formula. Conversely that formula makes conjugation by $\sum_v t_v p_v$ agree with σ on generators. Two such potentials have ratio constant along edges, hence constant by connectivity; normalization gives uniqueness. □

Theorem 5. *Every edge-diagonal action of a finite abelian group on $C^*(E)$, for E as above, is strongly approximately inner.*

Proof. Put $A = C^*(E)$, a unital Kirchberg algebra by the standard graph simplicity and pure-infiniteness criteria. First suppose the action is pointwise outer. For fixed g, choose real a_e with $e^{ia_e} = \chi_e(g)$. The assignments

$$\delta_t(p_v) = p_v, \qquad \delta_t(S_e) = e^{ita_e} S_e \quad (0 \le t \le 1)$$

preserve the graph relations and have inverse phases, so give automorphisms. They are point-norm continuous, first on generator polynomials and then by density. All commute with every α_h. They form an equivariant homotopy from id to α_g. Lemma 3 applies. No graph-period condition occurs.

For the general action let $K = \{k \in G : \alpha_k \text{ is inner}\}$, the kernel of the homomorphism $G \to \mathrm{Out}(A)$. Fix the paths θ_v in Lemma 4, once for all group elements, and define on all of G

$$D_g = \sum_v \chi_{\theta_v}(g)^{-1} p_v.$$

These unitaries form a genuine representation: $D_g D_h = D_{gh}$. They are fixed by every α_h. Lemma 4 gives $\alpha_k = \mathrm{Ad}(D_k)$ for $k \in K$. Thus

$$\beta_g = \mathrm{Ad}(D_g^*)\alpha_g$$

is an edge-diagonal action, trivial on K, with phases $\chi_{\theta_{s(e)}}(g)\chi_e(g)\chi_{\theta_{r(e)}}(g)^{-1}$. It descends to $H = G/K$. This quotient action is pointwise outer: if β_g is inner, then $\alpha_g = \mathrm{Ad}(D_g)\beta_g$ is inner, hence $g \in K$. The outer case proves strong approximate innerness of the H-action and therefore of its inflation β, with the same fixed algebra. If H is trivial, β is trivial and the unit implements every β_g exactly.

Finally $\beta_g(D_h) = D_h$ and $D_g D_h = D_{gh}$, so D is an honest β-cocycle and $\alpha_g = \mathrm{Ad}(D_g)\beta_g$. Lemma 2 proves the assertion with unitaries in the original A^α. This also covers $K = G$, where the D_g already implement α_g exactly in A^α, and covers nonfaithful actions. □

4 The original matrix presentation

Proof of Theorem 1. Write $\alpha_g(s_i) = \sum_j c_{ji}(g)s_j$. For $j \neq i$, multiply by p_j: the fixed range condition gives $p_j\alpha_g(s_i) = 0$, hence $c_{ji}(g)s_j = 0$. Thus $\alpha_g(s_i) = \chi_i(g)s_i$, where χ_i is a character by the action law. No simultaneous diagonalization is needed for this original definition. The broader domain-projection convention in [2] can instead be reduced to eigenvectors using its Proposition 1.8(2).

Let E have vertices $1, \ldots, n$ and an edge $i \to j$ whenever $M(i,j) = 1$. In $\mathcal{O}_M$ put $T_{ij} = s_i p_j$ for these edges. Then

$$T_{ij}^* T_{ij} = p_j, \qquad \sum_{j:M(i,j)=1} T_{ij} T_{ij}^* = p_i.$$

These give a graph Cuntz–Krieger family. Conversely the outgoing sums $s_i = \sum_{j:M(i,j)=1} T_{ij}$ in $C^*(E)$ satisfy the matrix relations; cross terms vanish since the edge initial projections p_j are orthogonal. The two formulas are inverse on generators. Thus $\mathcal{O}_M \cong C^*(E)$ equivariantly, with edge phases χ_i.

Essentiality gives at least one edge and both no sinks and no sources. In this finite setting, simplicity forces strong connectivity. Indeed, if the condensation graph has more than one component, choose a terminal component T and a different source component S. Since the original graph has no sources, S contains a cycle. The hereditary set T is nonempty and its saturated closure cannot absorb that cycle: at each stage, every still excluded cycle vertex has its next cycle edge outside the set. Its saturated closure is therefore proper. The associated nonzero proper graph ideal contradicts simplicity. This uses the standard saturated-hereditary ideal construction for graph algebras [9].

A directed cycle has algebra $M_m(C(\mathbb{T}))$, so is excluded by the hypotheses. Theorem 5 applies and proves the full assertion. □

Scope. All sequences obtained above lie exactly in the fixed algebra of the whole original action. The proof does not replace this condition by ordinary approximate innerness or by fixedness under a single group element. It uses neither matrix primitivity nor action faithfulness. The verification package preserves the longer return-corner deduction and the versioned source checks separately from this note; neither AI review nor those diagnostics certify novelty or replace the cited mathematical inputs.

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