A singleton-supported cycle correction to Byrnes's differential-poset bound

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Abstract

For every ordinary unweighted r-differential poset and $n \geq 2$, we subtract an explicit nonnegative term from Byrnes's cumulative upper bound for the nth rank size. The term is the cycle rank of the preceding incidence subgraph supported on elements with a singleton successor. Singleton witnesses transfer these cycles into the next incidence graph by replacing stars with trees having the same terminals. The correction can be positive and attained on standard reflected Young-prefix examples, and its coefficient one is sharp. A singleton-successor condition gives the conjectured local recurrence; two such successors give a conditional equality characterization. The result answers the printed request for an improvement of Byrnes's inequality in an incidence-dependent form. It does not prove the general local recurrence, improve the global Fibonacci envelope, or give a combinatorial proof of strict rank growth.

1 The improvement question and attribution

Majumder's problem in Oberwolfach Report 23/2018, printed p. 1452 [3], asks whether Byrnes's bound

$$p_n \leq \mathcal{U}_n := 1 + r \sum_{k=0}^{n-2} p_k + (r-1)p_{n-1} \tag{1}$$

can be improved toward Stanley's proposed local inequality

$$p_n \leq r p_{n-1} + p_{n-2}. \tag{2}$$

The same contribution separately asks for a combinatorial strict-growth proof. Our claim concerns the improvement request only. The correction uses incidences of the preceding ranks; no rank-number-only restriction is imposed in the printed request.

The operator framework, common-cover restrictions and reflection construction are classical [1]. Byrnes proves (1) on the way to the global Fibonacci envelope and states (2) separately [2, Chapter 5]. Incidence and hypergraph dimension sums already appear in [4, Remark 8]; the cycle identities below repackage these counts. Cycle-space monotonicity and terminal-preserving tree replacement are elementary graph facts. The substantive assertion proved here is the simultaneous transfer of the singleton-supported cycle rank, not a new general graph method.

2 Incidence graphs and exact slack

An ordinary r-differential poset is a locally finite $\mathbb{N}$-graded poset P with a unique minimum $\widehat{0}$, finite ranks P_j, and ordinary unweighted up and down operators over $\mathbb{Q}$ satisfying $DU - UD = rI$, where r is a positive integer. Write $p_j = |P_j|$, and let $d(x)$ and $u(x)$ count the lower and upper covers of x. Equivalently, the integer incidence counts satisfy

$$u(x) = d(x) + r, \qquad |\operatorname{pred}(x) \cap \operatorname{pred}(y)| = |\operatorname{succ}(x) \cap \operatorname{succ}(y)| \tag{3}$$

for distinct elements in a common rank. We do not interpret this identity only modulo a characteristic. The terminal equality to one in the OWR common-cover axiom is a typographical error: zero common covers are allowed under the standard definition [1, 5].

Lemma 2.1. *Distinct same-rank elements have at most one common lower cover and at most one common upper cover. Every consecutive-rank incidence graph is connected.*

Proof. If a pair has two common upper covers, (3) gives two common lower covers. This lower pair has the original pair as common upper covers. Iterating down the ranks contradicts the unique rank-zero element. The common lower count is equal to the upper count.

Let $\mathcal{B}_j$ be the bipartite incidence graph on $P_{j-1} \sqcup P_j$. The graph $\mathcal{B}_1$ is a star with r leaves. If $\mathcal{B}_j$ is connected, its projection on P_j, joining elements with a common predecessor, is connected (including the one-vertex case). By (3) each such adjacency has a common successor, so all P_j vertices lie in one component of $\mathcal{B}_{j+1}$. Every vertex of P_{j+1} has a lower cover, proving the induction. □

For a finite graph H, retaining isolated vertices, define $\beta(H) = |E(H)| - |V(H)| + c(H)$, where $c(H)$ is its component count. This is its cycle-space dimension over any fixed field; it is nonnegative and monotone under subgraph inclusion. Put $b_j = \beta(\mathcal{B}_j)$ for $j \geq 1$.

Proposition 2.2. *For $j \geq 1$, the number of edges of $\mathcal{B}_j$ is $e_j = r \sum_{k=0}^{j-1} p_k$, and*

$$b_j = r \sum_{k=0}^{j-1} p_k - p_{j-1} - p_j + 1, \tag{4}$$

$$p_n = \mathcal{U}_n - b_n \qquad (n \geq 2), \tag{5}$$

$$b_n - b_{n-1} = r p_{n-1} + p_{n-2} - p_n \qquad (n \geq 2). \tag{6}$$

Proof. Summing the degree identity gives $e_j = e_{j-1} + r p_{j-1}$, with $e_0 = 0$. Connectivity gives (4); rearranging and subtracting gives the remaining identities. □

Thus Byrnes's bound is $b_n \geq 0$, whereas (2) requires $b_n \geq b_{n-1}$. Nonnegative cycle rank alone does not give monotonicity.

3 The supported-cycle theorem

Fix $n \geq 2$ and set $Z = P_{n-2}$, $X = P_{n-1}$. Define

$$S = \{z \in Z : \text{some } a \in X \text{ has } \operatorname{pred}(a) = \{z\}\}.$$

Such an a is a *singleton successor* of z; it does not mean that z has only one successor. Let H_S retain all vertices of X, precisely S on the other side of $\mathcal{B}_{n-1}$, and all corresponding incidence edges. Set

$$q_n = \beta(H_S) = \sum_{z \in S} (u(z) - 1) - p_{n-1} + c(H_S). \tag{7}$$

Theorem 3.1. *For every ordinary r-differential poset and every $n \geq 2$,*

$$0 \leq q_n \leq b_{n-1}, \qquad b_n \geq q_n. \tag{8}$$

Consequently

$$p_n \leq 1 + r \sum_{k=0}^{n-2} p_k + (r-1)p_{n-1} - q_n, \tag{9}$$

$$p_n \leq r p_{n-1} + p_{n-2} + b_{n-1} - q_n. \tag{10}$$

The correction is determined by the two preceding ranks, before P_n is constructed, with no further hypothesis on the poset.

Proof. The first two inequalities follow from $H_S \subseteq \mathcal{B}_{n-1}$. For each $z \in S$, choose $a_z \in X$ with predecessor set $\{z\}$ and put $N_z = \operatorname{succ}(z) \subseteq X$. For every $y \in N_z \setminus \{a_z\}$, the pair a_z, y has unique common predecessor z, hence unique common successor $t \in P_n$. Every lower cover w of t lies in N_z: if $w \neq a_z$, its common successor with a_z forces a common predecessor, which must be z.

Take every nonsingleton upper vertex t covering a_z, with all its incidence edges, and all terminals N_z. Two different such t cannot share another terminal, by Lemma 2.1. Each terminal other than a_z occurs exactly once. Thus these stars, joined at a_z, form a tree T_z connecting precisely N_z. If $|N_z| = 1$, take the one-vertex tree. A rank-n internal vertex cannot belong to both T_z and T_w for $z \neq w$: it would cover the witnesses a_z, a_w, whose predecessor sets are disjoint, contrary to (3). Different trees can share only X terminals.

Let K be their union together with every vertex of X, including isolates. Replace each star of H_S centered at z by T_z. If T_z has a internal vertices, it has $|N_z| + a - 1$ edges. The replacement changes both vertex and edge counts by $a - 1$ and preserves the connections among terminals. Internal vertices from different replacements are disjoint, so components and cycle rank are preserved: $\beta(K) = \beta(H_S) = q_n$. Since $K \subseteq \mathcal{B}_n$, cycle-space inclusion gives $b_n \geq q_n$. Equations (5) and (4) now prove the bounds. □

4 When the local bound and equality follow

Corollary 4.1. *If every $z \in P_{n-2}$ has a singleton successor in P_{n-1}, then $p_n \le rp_{n-1} + p_{n-2}$. In particular this holds for every admissible next extension whenever rank $n-1$ was formed by reflection-extension.*

Proof. Here $S = Z$, so $q_n = b_{n-1}$. Reflection creating rank $n-1$ adds r singleton successors of every rank-$(n-2)$ element; see the construction in Section 5. The extension to rank n may be arbitrary. □

Theorem 4.2. *Suppose $n \ge 3$ and every $z \in P_{n-2}$ has at least two distinct singleton successors in P_{n-1}. Equality in the local bound holds if and only if the extension to rank n is reflection-extension, up to relabeling that rank.*

Proof. Choose one witness per z in the main proof. Now K is connected and contains all X vertices. Each unused upper vertex t attaches by its complete star to X, adding $d(t) - 1$ to cycle rank. Equality means $b_n = b_{n-1} = \beta(K)$, so every unused upper vertex is singleton.

For a fixed z, take distinct singleton successors $a = a_z, b$. Every nonsingleton upper vertex covering b belongs to K and specifically to T_z: if it belonged to T_w, its common predecessor with the witness a_w would force $w = z$. Every internal vertex of T_z covers a. Let h be the unique common successor of a, b. For each $y \in N_z \setminus \{b\}$, its common successor with b is nonsingleton, lies in T_z, and covers a. Uniqueness for a, b forces it to be h. Hence h covers all N_z, and no second nonsingleton internal vertex can share a pair of terminals.

Since $n \ge 3$, each z has a predecessor and $u(z) \ge r + 1 \ge 2$. There is therefore precisely one nonsingleton vertex h_z over each N_z. They are distinct by the witness argument. Each $x \in X$ belongs to exactly $d(x)$ of these neighborhoods; the degree identity leaves exactly r singleton successors over x. This is reflection-extension. Conversely reflection has $|Z| + r|X|$ upper vertices and gives equality. □

The following established lemma is [5, Proposition 3.4]; its short counting proof is included to make the consequence self-contained.

Lemma 4.3 (Gaetz–Venkataramana)**.** *Every rank-one element has at least two singleton successors.*

Proof. There are r rank-one elements, each with $r + 1$ successors. At a fixed z, nonsingleton successors use disjoint nonempty subsets of the other $r - 1$ rank-one elements; otherwise a pair would have two common successors. There are at most $r - 1$ nonsingleton successors, leaving at least two. □

Thus, for every $r \ge 1$, $p_3 \le rp_2 + r$, with equality precisely for reflection from rank two. This is a consequence of the transfer/equality theorem and the credited lemma, not a claim that the lemma is new.

5 Standard examples giving strictness and sharpness

We recall Wagner reflection, as presented in [1, Proposition 6.1 and the following paragraph]. Given a valid prefix through rank $m \geq 1$, add one vertex over the entire successor set in P_m of each element of P_{m-1}, and add r distinct singleton successors over each element of P_m. Each current x gets $d(x) + r$ successors; a current pair gets one common new successor for each common predecessor. Every new vertex has a predecessor and the rank is finite. Thus the prefix prolongs indefinitely to an ordinary differential poset. Its next rank has $p_{m-1} + rp_m$ elements. The constructions below use this classical operation and classical Young prefixes, rather than new poset families.

For $r = 3$, take rank one $\{a, b, c\}$. Put one rank-two element over each of $\{a, b\}, \{a, c\}, \{b, c\}$ and two singleton successors over each rank-one element. Each rank-one element has four successors and every pair has one. This is the prefix of Y^3 through rank two, with ranks $(1, 3, 9)$. Reflect to rank three and continue reflecting. Then

$$(p_0, p_1, p_2, p_3) = (1, 3, 9, 30), \qquad e_2 = 12, \quad e_3 = 39, \quad b_2 = b_3 = q_3 = 1.$$

Byrnes's rank-three bound is 31 and (9) gives the attained bound 30.

For $r = 1$, use Young's lattice Y through rank five and reflect twice, then continue reflecting. Young's lattice consists of integer partitions, ordered by addition of one box. Its first six rank sizes are $(1, 1, 2, 3, 5, 7)$, so the extended ranks are

$$(1, 1, 2, 3, 5, 7, 12, 19).$$

Here $e_6 = \sum_{k=0}^{5} p_k = 19$ and $p_5 + p_6 = 19$, hence $b_6 = 1$. Reflection at rank six gives a singleton successor for each rank-five element, so $q_7 = b_6 = 1$; reflection at rank seven gives $b_7 = 1$. Byrnes's bound 20 is reduced to the attained bound 19. The standard reflection argument proves infinite realization in both cases.

In either example $p_n = \mathcal{U}_n - q_n$ with $q_n = 1$. Replacing the coefficient one in (9) by any $c > 1$ would yield a false bound $p_n \leq \mathcal{U}_n - cq_n < p_n$. This proves coefficient sharpness, not optimality of every possible use of the preceding incidence data.

Boundary cases and the exact remaining gap

No b_0 is needed: $b_1 = 0$. At $n = 2$, $Z = \{\widehat{0}\}$ and every rank-one element is a singleton successor, so $q_2 = 0$ and $p_2 \leq r^2 + 1$. If also $r = 1$, the replacement tree has one terminal and no internal vertex. For empty S, H_S is the isolated vertex set X and $q_n = 0$.

Always $\mathcal{U}_n - (rp_{n-1} + p_{n-2}) = b_{n-1}$, so the refined right side lies between Byrnes's and the proposed local right sides. The unremoved allowance is exactly $b_{n-1} - q_n$. It can be positive: Young's lattice at $n = 6$ has $b_5 = 1, q_6 = 0$, refined bound 13, local bound 12, and actual $p_6 = 11$. The general recurrence still requires $b_n \geq b_{n-1}$. No combinatorial strict-growth argument follows here; the standard strict-growth statement also has the exception $r = n = 1$, when $p_0 = p_1 = 1$.

6 Verification, related work, and research status

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The checker reconstructs predecessor sets, verifies every integer commutator, connectivity, exact cycle identities and the actual witness-tree replacements. Its edge-clique partition enumeration gives labeled prefix counts $1, 1, 1, 1, 2, 7, 60$ for $r = 1$ through rank seven, $1, 1, 4$ for $r = 2$ through rank three, and $1, 2, 288$ for $r = 3$ through rank three. These are not isomorphism counts. It checks reflected prefixes for $1 \le r \le 5$ through rank five, Young's lattice through rank twelve, and both sharp examples. The 5764 witness selections use first and last choices in every reflection-only run, irrespective of product size. Enumerated-prefix, Young-lattice and separate sharp-example runs exhaust products of at most 2000 choices and use first/last above 2000. The proof establishes arbitrary witness choice.

Negative controls reject deletion of a top vertex and a false minimum degree, and refute subtracting $2q_n$ in the sharp examples. The relaxed layer sizes $(1, 3, 3, 7)$ satisfy the two middle commutators for $r = 1$ but violate the rank-zero commutator; they do not form a differential-poset prefix. Their pair-block projection has a triangle and their bipartite incidence graph a six-cycle. Normal, `-O`, and `-OO` runs agree byte for byte. These finite computations support implementation and example correctness; the universal theorem and infinite realization rest on the written proofs.

Singleton and crown techniques have earlier antecedents [6, 7]; the exact rank-one two-singleton lemma used here is [5, Proposition 3.4]. Reflection of arbitrary Young prefixes is already covered by [1]. Related upper-extremal investigations include Qing's 2008 thesis and Wright's 2019 slides/abstract [8, 9]; their global-envelope or restricted-prefix scope is distinct from the supported-cycle assertion. Byrnes also uses cycle arguments under other hypotheses, so this note asserts no new general use of cycles in the subject.

A dated bounded primary-source comparison, completed 22 October 2026, found no earlier equivalent supported-cycle upper-bound theorem in the inspected material. Historical firstness, first application and present worldwide openness are not certified. Source versions, inspected portions and access limits are recorded in the companion attribution dossier. The final publisher bodies of Gaetz–Venkataramana and Miller were not obtained; their versioned primary preprints were inspected. No subsequent full proof corresponding to Wright's talk was located, and subscription indexing and unpublished work remain coverage limits. The classification of this result concerns the literal printed improvement request, not every stronger rank question.

AI involvement, review, and licenses

AI tools were extensively involved in research planning, mathematical exploration and proof development/reconstruction, code and exact computations, literature and source investigation, writing, editing, and adversarial mathematical and priority auditing. Alec Kriebel is the named human author. This manuscript is unrefereed and has not undergone professional independent human peer review, journal acceptance or formal proof-assistant certification. Internal AI reviews do not supply those certifications. A fresh whole-package review is a separate release gate. Original prose is offered under CC BY 4.0; the companion code is MIT licensed. Cited works retain their respective rights.

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