Simple Young modules with nonprojective polynomial injective hulls

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Abstract

For every odd prime p, the Young module labelled by $\lambda = (2p-2, 1)$ in characteristic p is simple, whereas its corresponding polynomial injective hull $I_n(\lambda)$ is nonprojective for every $n \geq 2$. At parameter $q = 1$, this gives a negative answer to the unrestricted formulation of Miyachi's Conjecture 7 in the 2009 Oberwolfach report. The proof combines classical representation theory with a labelled rank-truncation argument. We include explicit proofs of the local calculations and reproducible supporting checks. Historical priority remains unresolved; no firstness claim is made.

1 The printed question

Let k be algebraically closed. In degree r, write $I_n(\nu)$ for the injective hull of the simple polynomial $\mathrm{GL}_n(k)$-module $L_n(\nu)$, equivalently its injective hull for the Schur algebra $S(n, r)$. At stable rank $N \geq r$, the Schur functor F defines $Y^\nu = \mathsf{F}I_N(\nu)$, the Young module in the convention of [1, pp. 945–946]. Here the parameter is $q = 1$, so the Hecke algebra is $k\mathfrak{S}_r$; in characteristic p, its quantum characteristic is p.

The first proposal in [1, Conjecture 7, p. 946] asks whether simplicity of $\mathrm{top}(Y^\nu)$ is equivalent to projectivity of $I_n(\nu)$ for *some* $n \geq \ell(\nu)$. The index-set criterion involving Λ^i_n is presented there as a stronger proposal. The characteristic-zero assumption in the subsequent subsection 0.5 concerns a different method for simple Specht modules; it is not attached to the displayed Conjecture 7.

Theorem 1. *Let p be any odd prime and let k be algebraically closed of characteristic p. Put $r = 2p-1$ and $\lambda = (2p-2, 1)$. Then Y^λ is simple of dimension $2p-2$, but $I_n(\lambda)$ is not projective for any integer $n \geq 2$.*

Thus $p = 3, \lambda = (4, 1)$ already refutes the broad printed proposal. The theorem does not address a characteristic-zero variant with a nontrivial root-of-unity parameter, nor the separate De Visscher–Donkin classification of projective-injective polynomial modules. In particular, being row p-regular is not being p-restricted (column p-regular), and no projectivity inference from row regularity is used.

2 Established structural inputs

We use the following standard facts from [2, Sections 2.2–2.4 and 3]. Page locators here refer to the manuscript dated 12 January 2005 (printed pp. 7–12), whose repository cover is not counted.

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Costandard reciprocity gives

$$(I_n(\nu) : \nabla_n(\mu)) = [\nabla_n(\mu) : L_n(\nu)]. \tag{1}$$

A nonzero right side requires μ to dominate ν. An indecomposable polynomial injective is projective only if it is contravariantly self-dual and an indecomposable tilting module. Conversely, an injective tilting module is projective. Contravariant duality fixes simple labels.

For $N \geq n$, exact rank truncation takes $I_N(\nu)$ to $I_n(\nu)$ and $T_N(\mu)$ to $T_n(\mu)$ when the respective labels have at most n parts. We apply these labelled statements, not preservation of projectivity for arbitrary modules under an idempotent functor. Finally, $\nabla_n(r) = \mathrm{Sym}^r(k^n)$ and $\nabla_2(a,b) = \det^b \otimes \mathrm{Sym}^{a-b}(k^2)$. The tensor products of symmetric powers are injective, with injective multiplicities given by the corresponding weight multiplicities of simples [2, Section 3, printed p. 12]. None of these inputs assumes either conjecture. The stable Schur conventions also appear in [3, printed p. 3].

3 The Young module and its label

Let $V = k^r$ be the natural permutation module and put $A = \{(a_1, \ldots, a_r) : \sum_i a_i = 0\}$. Since $p \nmid r$, the constant line splits off and $V = k \oplus A$. For a nonzero submodule $U \subseteq A$, choose $0 \neq v \in U$. Its coordinates cannot all agree. If $v_i \neq v_j$, then

$$v - (ij)v = (v_i - v_j)(e_i - e_j) \in U.$$

Permuting the coordinates supplies all differences $e_i - e_j$, which span A. Hence A is simple of dimension $r - 1$. This is the classical natural permutation/augmentation calculation.

For clarity, we check the label in the source's $\mathsf{F}I$ convention. Take $N \geq r$, $E_N = k^N$ and $H = \mathrm{Sym}^{r-1} E_N \otimes E_N$. Its injective multiplicities are the $(r-1, 1, 0, \ldots, 0)$-weight multiplicities of simples. Dominance leaves only labels (r) and λ, each with multiplicity one. For λ this is the highest weight. For $L_N(r)$, the weight occurs in the simple socle $L_N(r)$ of $\nabla_N(r) = \mathrm{Sym}^r E_N$, which contains x_1^r: the coefficient of t in $(x_1 + tx_2)^r$ is $rx_1^{r-1}x_2 \neq 0$. The ambient weight space is one-dimensional. Thus

$$H \cong I_N(r) \oplus I_N(\lambda).$$

Reciprocity at the maximal label gives $I_N(r) = \mathrm{Sym}^r E_N$. Multiplication $m : H \to \mathrm{Sym}^r E_N$ splits: the polarization map

$$\delta(f) = \sum_i \frac{\partial f}{\partial x_i} \otimes x_i \quad \text{satisfies} \quad m\delta = r \,\mathrm{id}.$$

Cancellation of indecomposable summands yields $\ker m \cong I_N(\lambda)$. On multilinear weight spaces, H has basis $(\prod_{j \neq i} x_j) \otimes x_i$ for $1 \leq i \leq r$, and m is coordinate summation onto $k \prod_j x_j$. Exactness of F therefore gives $\mathsf{F}I_N(\lambda) \cong A$, as required. The stable Schur functor is never applied at rank $2 < r$.

4 The rank-two obstruction

Write $E = k^2$, with basis x, y. In a fixed symmetric power, the monomials $v_i = x^{m-i}y^i$ have distinct algebraic-torus weights. An invariant subspace contains individual weight components. It also contains every coefficient in t of a unipotent translate: Vandermonde inversion over the infinite field k extracts those coefficients from evaluations. This is a statement about rational $\mathrm{GL}_2(k)$-modules, not just the finite group $\mathrm{GL}_2(\mathbb{F}_p)$.

Lemma 1. *The module* $\mathrm{Sym}^{2p-1} E$ *is simple. The module* $B = \mathrm{Sym}^{2p-3} E$ *is a nonsplit extension of two distinct simple modules: its socle has highest weight* $(2p-3, 0)$ *and dimension* $2p-4$*; its top has highest weight* $(p-1, p-2)$ *and dimension* 2*.*

Proof. From any monomial, the substitution $y \mapsto y+tx$ extracts the highest monomial. Translating that highest monomial in the opposite direction extracts the binomial-coefficient multiples of all monomials. For degree $2p-1$, every coefficient is nonzero, since in characteristic p

$$(1+t)^{2p-1} = (1+t^p)(1+t)^{p-1},$$

and the coefficients of $(1+t)^{p-1}$ are alternating units. This proves the first assertion.

For $m = 2p-3$, let

$$W = \mathrm{span}\{v_i : 0 \le i \le p-3 \text{ or } p \le i \le 2p-3\} \subset B.$$

It is the image of the injective equivariant multiplication $E^{(1)} \otimes \mathrm{Sym}^{p-3} E \to B$, where $E^{(1)}$ denotes Frobenius twist, via $x^{(1)} \otimes f \mapsto x^p f$ and $y^{(1)} \otimes f \mapsto y^p f$. The two monomial ranges are disjoint. The same highest-monomial argument, now using

$$(1+t)^{2p-3} = (1+t^p)(1+t)^{p-3},$$

shows that every nonzero submodule of W contains all its displayed monomials. Hence W is simple with the stated highest weight and dimension.

The quotient has the two weight lines represented by v_{p-2}, v_{p-1}. The opposite unipotents connect them with coefficient $p-1 \neq 0$, so the quotient is simple of highest weight $(p-1, p-2)$. Any invariant complement to W must be a sum of torus weight spaces, hence must equal $\mathrm{span}(v_{p-2}, v_{p-1})$. But the translate of v_{p-2} under $y \mapsto y+tx$ has the nonzero coefficient $x^{2p-3} \in W$ at t^{p-2}. There is no complement. A nonsplit extension of two distinct simples has precisely the asserted socle and top. □

These rank-two structures are established representation theory; for published formulas yielding them by duality and determinant twist, see [4, Proposition 3.1, classical analogue, and Theorem 3.3]. The explicit proof makes their use here checkable.

Only (r) strictly dominates $\lambda = (r-1, 1)$. By Lemma 1, $\nabla_2(r) = \mathrm{Sym}^r E$ is simple and does not contain $L_2(\lambda)$. Equation (1) therefore gives

$$I_2(\lambda) = \nabla_2(\lambda) = \det \otimes B.$$

Its socle and top are $L_2(2p-2, 1)$ and $L_2(p, p-1)$, respectively. Their highest weights differ, including when $p = 3$ and both dimensions are 2. Thus $I_2(\lambda)$ is not contravariantly self-dual and is not projective.

5 Excluding every permitted rank

Suppose $I_N(\lambda)$ were projective for some $N \ge 2$. It would be $T_N(\mu)$ for a partition μ of r. Since $L_N(\lambda)$ is a composition factor, μ dominates λ. Consequently $\mu = (r)$ or λ; both labels have at most two parts. Labelled rank truncation gives

$$I_2(\lambda) \cong T_2(\mu).$$

This would be an injective tilting module, hence projective, contradicting the rank-two calculation. This proves Theorem 1 for every $N \ge 2$.

Corollary 1. *The stronger simple-top/Λ_n^i criterion in the same printed Conjecture 7 is false in this setting.*

Proof. Membership in the constructed Λ_n^i family is already a sufficient condition for $I_n(\lambda)$ to be projective [2, 1]. Theorem 1 excludes membership at every allowed rank. No completeness assertion about that family is needed. □

6 Verification, attribution and status

The companion archive preserves the frozen candidate, independent mathematical and priority reviews, source identities and reproducible exact checks. Formal polynomial-coefficient calculations in characteristics 3 and 5 verify the local constituents and nonsplitting. A separate tensor-orbit computation constructs all 56 basis elements of $S(2,5)$, checks its 3136 products and independently finds $\dim P_2(3,2) = 6$, whereas $\dim I_2(4,1) = 4$. That optional computation uses NumPy; the core checks use Python's standard library. Finite computations support the argument, and do not prove its all-prime or all-rank assertions.

The Young/augmentation calculation, rank-two structure and Schur-algebra inputs are classical. The note's purpose is their explicit assembly into a counterexample to the broad printed 2009 proposal. An extensive but bounded literature audit on 16 October 2026 did not verify an earlier complete bridge refutation. Priority remains unresolved: some older two-row papers, tables and later endomorphism-algebra papers were not available in full, and citation coverage is incomplete. No claim of firstness, certified novelty, or current worldwide open status is made. The source archive records exact read scopes and gaps.

AI tools were used extensively in developing and checking the argument, literature investigation, computations, drafting, reproduction and adversarial review. This is an unrefereed preprint without external human peer review or proof-assistant certification. The note and authored companion material are licensed under CC BY 4.0.

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