

A simple brace outside asymmetric products of two trivial braces

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Abstract

We give a negative answer to Cedó's Problem 9 in Oberwolfach Report 51/2019. An explicit left brace of order 20 000 is nontrivial and simple, has metabelian multiplicative group with abelian Sylow subgroups, and is not an asymmetric product of two trivial braces. The obstruction is intrinsic: every such two-factor product has zero triple star products within each additive Sylow subbrace, whereas the constructed brace has $(x*x)*x = x^3 \neq 0$ in its 5-primary subbrace. The obstruction covers the full admissible symmetric-cocycle definition, not only a chosen bilinear construction family. All operations and the simplicity argument are given directly; exact verification programs and independent adversarial audits accompany the note.

1 The question

A left brace $(B, +, \circ)$ has an abelian additive group, a multiplicative group, and the compatibility

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

Write $\lambda_a(b) = -a + a \circ b$ and $a * b = \lambda_a(b) - b$. The maps λ_a are additive automorphisms and satisfy $\lambda_{a \circ b} = \lambda_a \lambda_b$. The brace is trivial when $a * b = 0$ for all a, b . An ideal is an additive subgroup invariant under all λ_a and normal in the multiplicative group; a nonzero brace is simple if its only ideals are 0 and itself.

Cedó asks whether every finite nontrivial simple left brace with a metabelian multiplicative group and abelian multiplicative Sylow subgroups is an asymmetric product of two trivial braces [2, p. 3228, Problem 9]. We retain all these hypotheses.

Theorem 1. *There is a nontrivial simple left brace B of order 20 000 with*

$$(B, +) \cong C_5^4 \times C_2^5, \quad (B, \circ) \cong (C_5^3 \rtimes_{\tau} C_2) \times (C_2^4 \rtimes_M C_5),$$

where τ negates the first and third coordinates of $\mathbb{F}_5^3$ and M is the five-cycle on the even-weight subspace of $\mathbb{F}_2^5$. Its multiplicative group is metabelian with every Sylow subgroup abelian. No brace isomorphic to B is an asymmetric product of two trivial braces.

2 An invariant of every two-trivial-factor product

We use the full asymmetric-product definition of Catino–Colazzo–Stefanelli, in its left-brace formulation [3, 4, 6]. With trivial factors T, S , an admissible presentation has operations

$$(t, s) + (u, v) = (t + u, s + v + b(t, u)), \tag{1}$$

$$(t, s) \circ (u, v) = (t + \alpha_s(u), s + v), \tag{2}$$

where $\alpha : S \rightarrow \text{Aut}(T, +)$ is a group homomorphism. Initially $b : T \times T \rightarrow S$ is only a normalized symmetric 2-cocycle: $b(t, 0) = 0$, $b(t, u) = b(u, t)$, and

$$b(t + u, w) + b(t, u) = b(t, u + w) + b(u, w).$$

Admissibility means that these operations define a left brace, as in the cited general definition.

Lemma 2. *If a finite brace has a presentation (1)–(2) with T, S trivial and all its multiplicative Sylow subgroups abelian, then*

$$(a * b) * c = 0$$

for all a, b, c in any one additive Sylow subbrace.

Proof. The cocycle identity gives

$$(t, s) - (u, v) = (t - u, s - v - b(t - u, u)),$$

and therefore, before assuming biadditivity,

$$\lambda_{(t,s)}(u, v) = (\alpha_s(u), v - b(\alpha_s(u), t)).$$

Since $(t, 0) \circ (u, 0) = (t + u, 0)$, lambda composition evaluated on $(w, 0)$ forces $b(w, t + u) = b(w, t) + b(w, u)$. Symmetry makes b biadditive.

For a prime p , write T_p, S_p for the primary subgroups of the finite abelian factor groups. Mixed-primary values of b vanish, and $b(T_p, T_p) \subseteq S_p$: each value is killed by the relevant prime powers. Consequently the set $P = T_p \times S_p$ is an additive subgroup of the full p -part of the order. Its additive law need not split, especially for $p = 2$; its order suffices. The action preserves T_p , so the same set is also a multiplicative Sylow subgroup.

Commuting $(t, 0)$ and $(0, s)$ inside this abelian multiplicative Sylow subgroup gives $\alpha_s(t) = t$ for $s \in S_p, t \in T_p$. For $a = (t, s)$ and $b_0 = (u, v)$ in P , twisted additive subtraction now gives

$$a * b_0 = (u, v - b(u, t)) - (u, v) = (0, -b(u, t)).$$

Every $(0, w)$ with $w \in S_p$ has identity lambda action on P . Thus $(a * b_0) * c = 0$ for every $c \in P$. $\square$

The lemma does not require simplicity or metabelianity. Brace isomorphisms preserve additive Sylow subgroups and both star operations, so a nonzero primary triple star excludes every possible presentation, including one with different factors or coordinates.

3 The explicit brace

Let $R = x\mathbb{F}_5[x]/(x^5)$, with basis x, x^2, x^3, x^4 . In the ambient unital truncated polynomial ring, define

$$L(r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4}, \quad E(t) = t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}.$$

All denominators are invertible in $\mathbb{F}_5$. Formal expansion through degree four, with higher total degrees zero, proves

$$L(E(t)) = t, \quad E(L(r)) = r, \quad L(r + u + ru) = L(r) + L(u). \quad (3)$$

There is no division by five or infinite-series assumption. Let σ be substitution $x \mapsto -x$, and put

$$\eta(r) = [x^4]L(r).$$

Then $\eta(r + u + ru) = \eta(r) + \eta(u)$, $\eta(\sigma r) = \eta(r)$, and $L(\sigma r) = \sigma L(r)$.

Let V be the even-weight subspace of $\mathbb{F}_2^5$. For $v, w \in V$ set

$$\beta(v, w) = \sum_{i=0}^4 v_i w_i, \quad Q(v) = \frac{\text{wt}(v)}{2} \pmod{2}.$$

The integer weight identity proves

$$Q(v + w) = Q(v) + Q(w) + \beta(v, w). \quad (4)$$

The restricted dot form β is nondegenerate: the orthogonal complement of the even-weight space in $\mathbb{F}_2^5$ consists of constant vectors, and its only even-weight constant vector is zero. Cyclic rotation M preserves V, β, Q and has order five. Its fixed vectors in V are zero, so $M - \text{id}$ is invertible.

Take $B = R \times V \times \mathbb{F}_2$ with componentwise addition. For $a = (r, v, z)$ write $s(a) = \eta(r)$ and $e(a) = z + Q(v)$, and define

$$(r, v, z) \circ (u, w, Z) = (r + \sigma^{e(a)}u + r\sigma^{e(a)}u, v + M^{s(a)}w, z + Z + \beta(v, M^{s(a)}w)). \quad (5)$$

Exponents of M are interpreted modulo five.

Proposition 3. *These operations make B a left brace with the two groups stated in Theorem 1.*

Proof. Write $L(r) = l_1x + l_2x^2 + l_3x^3 + sx^4$ and $k = (l_1, l_2, l_3)$. The map

$$\Psi(r, v, z) = ((k, z + Q(v)), (v, s))$$

is a bijection onto $(\mathbb{F}_5^3 \rtimes_{\tau} C_2) \times (V \rtimes_M C_5)$: recover r with E , and then z . For a second input with corresponding coordinates h, f, w, t , (3) gives the output logarithm $L(r) + \sigma^{e(a)}L(u)$. Its first three coefficients are $k + \tau^{e(a)}h$ and its fourth is $s + t$. Equation (4) gives

$$z + Z + \beta(v, M^s w) + Q(v + M^s w) = z + Q(v) + Z + Q(w) = e(a) + f.$$

Thus the complete transformed product is exactly $((k + \tau^{e(a)}h, e(a) + f), (v + M^s w, s + t))$. This proves the group law, including associativity and inverses.

Direct additive subtraction in (5) yields

$$\lambda_{(r,v,z)}(u, w, Z) = ((1 + r)\sigma^{z+Q(v)}u, M^{\eta(r)}w, Z + \beta(v, M^{\eta(r)}w)). \quad (6)$$

The first block is an invertible $\mathbb{F}_5$ -linear map, since $1 + r$ is a unit. The last two coordinates form an invertible triangular $\mathbb{F}_2$ -linear map. Every lambda map is therefore an additive automorphism, proving the left-brace identity. Finally $|B| = 5^4 2^4 2 = 20\,000$. $\square$

The commutator subgroup of the displayed multiplicative group is

$$(\tau - \text{id})\mathbb{F}_5^3 \times (M - \text{id})V \cong C_5^2 \times C_2^4,$$

which is abelian of order 400. Both actions are nontrivial, so the group is nonabelian and metabelian. Explicit Sylow subgroups are $\mathbb{F}_5^3 \times C_5$ and $C_2 \times V$ of orders 625 and 32, respectively. Both are abelian, and Sylow conjugacy covers all Sylow subgroups. No normal-Sylow assumption is used.

4 Simplicity and exclusion

Proposition 4. *Every nonzero ideal of B is B .*

Proof. For an ideal I , $i \in I$ and $g \in B$, both $\lambda_g(i) - i$ and $i * g$ lie in I . The second inference follows in the quotient brace B/I , where i becomes zero; it is not an assumption about arbitrary left ideals.

If a nonzero element $(r, v, z) \in I$ has $r \neq 0$, its additive sixfold multiple is $(r, 0, 0) \neq 0$. Let cx^j be the lowest nonzero term of r . For $j < 4$, formula (6) gives

$$(\lambda_{(x^{4-j}, 0, 0)} - \text{id})(r, 0, 0) = (cx^4, 0, 0).$$

For $j = 4$ this already holds without applying a lambda difference. Additive rescaling gives $a = (x^4, 0, 0) \in I$.

Otherwise I has a nonzero element $(0, v, z)$. If $v = 0$ it is $b = (0, 0, 1)$. If $v \neq 0$, choose w with $\beta(w, v) = 1$. Then $(\lambda_{(0, w, 0)} - \text{id})(0, v, z) = b$. Thus every nonzero ideal contains a or b .

From b one obtains $b * (x, 0, 0) = (3x, 0, 0)$ and hence a nonzero ring element, which gives a . Conversely, $\eta(x^4) = 1$ implies

$$a * (0, w, 0) = (0, (M - \text{id})w, 0).$$

Surjectivity of $M - \text{id}$ puts every pure V vector into I , and a binary lambda difference gives b . Thus either starting point gives all of $P_2 = \{0\} \times V \times \mathbb{F}_2$ and also $(x, 0, 0)$. For $j = 1, 2, 3$,

$$(\lambda_{(x, 0, 0)} - \text{id})(x^j, 0, 0) = (x^{j+1}, 0, 0).$$

These monomials generate $P_5 = R \times \{0\} \times \{0\}$ additively. Hence I contains both primary parts and equals B . $\square$

On P_5 , star multiplication is ordinary ring multiplication. In particular $x * x = x^2 \neq 0$, so the brace is nontrivial, and

$$(x * x) * x = x^3 \neq 0.$$

Lemma 2 rules out every asymmetric product of two trivial braces. Together with Propositions 3 and 4, this proves Theorem 1 and gives the full negative answer to the original question.

5 Prior work, verification, and review status

Asymmetric and matched products, radical-ring braces, truncated logarithm/exponential identities and cyclic orthogonal actions are established methods [3, 4, 5, 6]. General matched-product formulas [1] recover (5), but their existence statement does not replace the all-ideal simplicity proof. No new general construction formalism is claimed. Recent work also uses local commutative algebras and invariant homomorphisms [7]; those ingredients themselves are not a novelty claim.

A literature audit on 17 October 2026 found no prior full answer or verified published full implication for this target. This is a bounded finding, not historical-priority certification. It included the current arXiv version of [8] and the recent constructions [9, 10]. The full bodies of [11, 12] and several final journal versions, including that of [8], were unavailable; their broader consequences are not excluded by abstracts. The companion audit records exact versions, inspection boundaries, access failures and citation-coverage limits.

The source/verification archive contains the original full proof, independently authored adversarial reports, four exact programs and expected results. Factored identities check the full construction; independent derivations from all 19 999 nonzero ideal seeds end at the nine additive generators. The archive retains deterministic seed receipts, sample derivations, deliberately broken controls and ordinary/optimized execution records. These supplement the universal proofs; no enumeration of all $20\,000^3$ triples is asserted. The universal nonpresentation lemma is proved above, rather than inferred from finite examples.

AI tools were used extensively in construction, proof development, computation, adversarial verification, literature research and writing. This version is **unrefereed**. AI adversarial review is not external human peer review, journal acceptance or formal proof-assistant certification. The note and authored supporting materials are released under CC BY 4.0.

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