Sharp modulus range for triangle counts in dense random graphs

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Abstract

Fix $0 < p < 1$, and let T_n count each triangle once in $G(n,p)$. For every positive integer sequence q_n, we give the explicit deduction

$$\max_{0\le a<q_n} |q_n \mathbb{P}(T_n \equiv a \pmod{q_n}) - 1| \longrightarrow 0 \quad \Longleftrightarrow \quad q_n = o(n^2).$$

Sufficiency follows from the quantitative local limit theorem of Sah and Sawhney, concentration, and finite-window summation. Necessity follows from an elementary edge-projection central limit theorem and a modular Fourier coefficient. This covers arbitrary, possibly oscillating or composite, modulus sequences. We also reconcile density and Gaussian unit-cell conventions. The result is a credited corollary of earlier local limit work, addressing the sharp range in Scott's 2009 report of joint work with Tateno. No novelty, historical firstness, or new local limit theorem is claimed.

1 Question, conventions, and result

Scott's report of joint work with Tateno [3, pp. 1279–1280] asks how far equidistribution of the triangle count modulo q extends. It announces the relative, uniform-in-residue conclusion for $q = o(n^2/\log n)$, without a primality restriction, already identifies $o(n^2)$ as the necessary scale, and announces a triangle local limit theorem. Here we spell out the complete sharp-range deduction from a later quantitative local limit theorem. This is a present-day deduction from prior results, not an assertion that this corollary was explicitly stated in an earlier source or that the question remained open until this note. The accompanying bounded literature audit is not a certificate of novelty.

The triangle local limit theorem was established by Gilmer and Kopparty [1], and quantitative pointwise estimates were obtained by Berkowitz [2]. These are substantive predecessors of the connected-count theorem used below, rather than new local limit results of this note. In fact, Gilmer–Kopparty's earlier characteristic-function bounds [1, Lemmas 4–5 in the 2014 manuscript] already imply the full sufficient modular range by summation on the nonzero modular Fourier grid. The supporting material gives that deduction; we do not assert that the corollary was explicitly printed there. Our main proof uses the clean quantitative local theorem for a transparent probability-mass argument.

Throughout, $p \in (0,1)$ is fixed, the edges of $G(n,p)$ are independent, and T_n is the number of unordered three-vertex sets spanning a triangle. Thus each triangle is counted once; a labeled-embedding count would be $6T_n$ and would have different modular restrictions. Write

$$\mu_n = \mathbb{E}T_n, \qquad \sigma_n^2 = \operatorname{Var} T_n, \qquad \Delta_n(q) = \max_{0\le a<q} |q\mathbb{P}(T_n \equiv a \pmod{q}) - 1|.$$

All asymptotic statements are as $n \to \infty$, and implicit constants may depend on fixed p. Standard-deviation normalization starts at $n \geq 3$.

Theorem 1. *For every sequence of positive integers q_n,*

$$\Delta_n(q_n) \longrightarrow 0 \quad \Longleftrightarrow \quad q_n = o(n^2).$$

No monotonicity, primality, or lower growth condition on q_n is required.

In particular $q_n = 1$ is permitted. Sparse or varying p, general subgraph counts, and induced-copy analogues are outside this theorem. The broader final conjecture in [3] concerns arbitrary fixed subgraphs and is a separate question. Its unrestricted Gaussian local limit part already has the ordinary $2K_2$ counterexample of Sah and Sawhney [4, Theorem 1.3]; that known obstruction is not a new result of this note.

2 Moments and the imported local theorem

Let I_e be the edge indicators, $\xi_e = I_e - p$, and $N = \binom{n}{2}$. Expanding the three factors for each triangle gives

$$T_n - \mu_n = L_n + R_n, \qquad L_n = p^2(n-2)\sum_e \xi_e, \tag{1}$$
$$R_n = p \sum_{\{e,f\} \text{ adjacent}} \xi_e\xi_f + \sum_{\{e,f,g\} \text{ triangle}} \xi_e\xi_f\xi_g.$$

Each unordered adjacent edge pair occurs once. Distinct edge sets index orthogonal centered products, since an edge in their symmetric difference has a centered factor occurring only once. There are $3\binom{n}{3}$ adjacent pairs. Consequently, with $\alpha = p^5(1-p) > 0$ and $D = N(n-2)^2$,

$$\begin{aligned} \mu_n &= \binom{n}{3}p^3, \\ \sigma_n^2 &= \alpha D + 3\binom{n}{3}p^4(1-p)^2 + \binom{n}{3}p^3(1-p)^3 \\ &= \binom{n}{3}p^3(1-p^3) + 12\binom{n}{4}p^5(1-p) \sim \frac{\alpha}{2}n^4. \end{aligned} \tag{2}$$

The coefficient $12\binom{n}{4}$ counts ordered pairs of distinct triangles sharing an edge. Thus

$$\sigma_n^2 \geq \alpha D, \qquad \sigma_n \sim \sqrt{\alpha/2}\, n^2. \tag{3}$$

The empty graph and a graph consisting of exactly one triangle and isolated vertices have positive probability, so the integer span is one. We do not assume every intermediate count is attainable.

Our substantive external premise is Sah–Sawhney's quantitative local limit theorem [4, Theorem 1.1], in the inspected arXiv:2006.11369v1. For ordinary connected subgraph counts it uses the actual mean and standard deviation in $G(n,p)$. Its ordinary-copy definition and independent-edge transfer are in Sections 4.1 and 5. Specializing to K_3, and choosing $\lambda = \min(p, 1-p)/2$ in its density hypothesis, gives, for every fixed $\varepsilon > 0$,

$$\sup_{k \in \mathbb{Z}} |\mathbb{P}(T_n = k) - g_n(k)| \leq \frac{\delta_n}{\sigma_n}, \qquad g_n(x) = \frac{1}{\sigma_n}\phi\Big(\frac{x - \mu_n}{\sigma_n}\Big), \quad \delta_n = C_{p,\varepsilon}n^{-1/2+\varepsilon}, \tag{4}$$

where $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$. The theorem counts each triangle once, rather than its six embeddings. We import this established, deep Fourier/decoupling theorem; we do not reprove or independently certify its entire proof. Only its uniform pointwise bound is needed. Fix $0 < \varepsilon < 1/2$, so $\delta_n\sqrt{\log n} \to 0$. An unquantified pointwise error cannot simply be summed over the full count support to obtain our relative modular assertion.

3 Sufficiency: finite-window summation

Finite-window, tail, and Gaussian-progression comparisons have classical precedents; see Weber [5, Theorem 1.4 and its proof in the 2022 preprint] for independent integer sums. We prove the required transfer estimates below; we do not apply that independent-sum theorem or its constants to the dependent triangle count.

Changing one edge changes T_n by at most $n-2$. The independent-edge exposure martingale and the bounded-range exponential-moment inequality give

$$\mathbb{P}(|T_n - \mu_n| \geq u) \leq 2e^{-2u^2/D}. \tag{5}$$

Indeed, each martingale increment has conditional range length at most $n-2$; its centered exponential moment is at most $e^{t^2(n-2)^2/8}$. Multiplying these bounds, applying exponential Markov, and optimizing $t = 4u/D$ proves each one-sided bound and hence (5).

Set

$$B_n = \sqrt{(16/\alpha)\log n}.$$

By (3) and (5), the probability outside $|T_n - \mu_n| \leq B_n\sigma_n$ is at most $2n^{-32}$. For large n, $\sigma_n \geq 1$ and $B_n \geq 4$. Comparing each Gaussian grid point on a tail with its adjacent interval toward the origin gives

$$\begin{aligned}\sum_{\substack{k\in\mathbb{Z}\\ |k-\mu_n|>B_n\sigma_n}} g_n(k) &\leq 2\int_{B_n-1/\sigma_n}^{\infty} \phi(x)\,dx \\ &\leq 2e^{-(B_n-1)^2/2} \leq 2e^{-B_n^2/4} = 2n^{-4/\alpha} \leq 2n^{-16}.\end{aligned} \tag{6}$$

Here $\alpha \leq 1/4$, and $(B-1)^2/2 \geq B^2/4$ for $B \geq 4$. These are full tails, so they bound each residue's omitted mass as well.

Fix $1 \leq q \leq \sigma_n$ and a residue a. There are at most $2B_n\sigma_n/q + 1$ integers in $a + q\mathbb{Z}$ inside the central interval. Use (4) only on those finitely many integers, and use the two full tails elsewhere. We obtain

$$q\left|\mathbb{P}(T_n \equiv a \pmod q) - \sum_{k\in a+q\mathbb{Z}} g_n(k)\right| \leq \delta_n(2B_n + q/\sigma_n) + 2q(n^{-32} + n^{-16}). \tag{7}$$

All factors of q are retained.

We also use the elementary shifted-mesh inequality

$$\left|h\sum_{j\in\mathbb{Z}} f(b+jh) - \int_{\mathbb{R}} f(x)\,dx\right| \leq h\int_{\mathbb{R}} |f'(x)|\,dx \qquad (h>0,\ b\in\mathbb{R}), \tag{8}$$

for absolutely continuous f with $f, f' \in L^1(\mathbb{R})$. To prove it, compare $hf(b+jh)$ with the integral over $[b+jh, b+(j+1)h]$; the difference is bounded by h times the integral of $|f'|$ over that interval.

Summation proves the bound. Applying the same comparison to $|f|$ justifies absolute convergence. For ϕ, $\int \phi = 1$ and $\int |\phi'| = \sqrt{2/\pi}$. Taking $h = q/\sigma_n$ and $b = (a - \mu_n)/\sigma_n$ in (8) yields

$$\left| q \sum_{k \in a+q\mathbb{Z}} g_n(k) - 1 \right| \le \sqrt{2/\pi}\, q/\sigma_n.$$

Together with (7), this proves the explicit uniform bound

$$\Delta_n(q) \le \delta_n(2B_n + q/\sigma_n) + 2q(n^{-32} + n^{-16}) + \sqrt{2/\pi}\, q/\sigma_n, \qquad 1 \le q \le \sigma_n. \tag{9}$$

If $q_n = o(\sigma_n)$, the bound applies eventually and every term tends to zero, since $\sigma_n = O(n^2)$. This proves sufficiency even when q_n/σ_n tends to zero arbitrarily slowly.

4 Necessity and the original cell convention

For sharpness, the local limit theorem is unnecessary. From (1)–(2), $\operatorname{Var} L_n = \alpha D = \Theta_p(n^4)$ and $\operatorname{Var} R_n = O_p(n^3)$. The binomial central limit theorem for $\sum_e I_e$, together with $R_n/\sigma_n \to 0$ in L^2, gives

$$Z_n = (T_n - \mu_n)/\sigma_n \implies N(0,1), \qquad \mathbb{E}|Z_n| \le 1.$$

Suppose $\Delta_n(q_n) \to 0$ but $q_n/\sigma_n \not\to 0$. On an infinite subsequence $q_n/\sigma_n \ge c > 0$; passing to a further subsequence, $r_n = \sigma_n/q_n \to r \in [0, 1/c]$. Eventually $q_n \ge 2$, so the sum of the q_n-th roots of unity is zero and

$$\left| \mathbb{E} e^{2\pi i T_n/q_n} \right| \le \sum_{a=0}^{q_n-1} |\mathbb{P}(T_n \equiv a \pmod{q_n}) - 1/q_n| \le \Delta_n(q_n) \longrightarrow 0. \tag{10}$$

Removing the centering phase leaves $\left|\mathbb{E} e^{it_n Z_n}\right|$, where $t_n = 2\pi r_n \to 2\pi r = t$. The varying frequency is legitimate because

$$\left| \mathbb{E} e^{it_n Z_n} - \mathbb{E} e^{itZ_n} \right| \le |t_n - t|\, \mathbb{E}|Z_n| \le |t_n - t| \longrightarrow 0.$$

The central limit theorem therefore makes the left side of (10) tend to $e^{-2\pi^2 r^2} > 0$, including 1 when $r = 0$, a contradiction. Thus $q_n = o(\sigma_n)$, equivalently $q_n = o(n^2)$. This subsequence argument includes nonmonotone, critical, and supercritical sequences, and completes the proof of Theorem 1.

For comparison with the original local theorem in [3], define its centered Gaussian unit-cell mass

$$c_n(k) = \int_{(k-\mu_n-1/2)/\sigma_n}^{(k-\mu_n+1/2)/\sigma_n} \phi(x)\, dx.$$

Taylor's theorem about $(k - \mu_n)/\sigma_n$, integrated symmetrically, cancels the linear term and proves, uniformly in k,

$$|c_n(k) - g_n(k)| \le \frac{\|\phi''\|_\infty}{24\sigma_n^3}.$$

Hence (4) also gives

$$\sup_{k \in \mathbb{Z}} |\mathbb{P}(T_n = k) - c_n(k)| = O_{p,\varepsilon}(n^{-5/2+\varepsilon}) + O_p(n^{-6}) = o(n^{-2}) \qquad (0 < \varepsilon < 1/2).$$

Thus the density convention of the imported theorem also supplies the original cell convention; this is not a change of the triangle target.

Verification and disclosure. The supporting materials distinguish fresh exact finite checks, proof reviews, and source inspection from historical records. Finite enumeration checks the moment and counting identities; it does not establish an asymptotic theorem. AI tools were used extensively in developing, reconstructing, drafting, and verifying this work, including independent internal AI adversarial reviews. This note is unrefereed and has received no external human peer review or formal proof-assistant certification. The imported local limit theorem remains a named external mathematical premise. No novelty or historical-priority claim is made.

References

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