

A seven-hyperplane counterexample to a generic-initial degree bound

Alec Kriebel

Independent researcher

ORCID: 0009-0001-9320-500X

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Abstract

For the characteristic-zero arrangement defined by $Q = xy(x - y)(x - 2y)zw(x + z + w)$, the degree-reverse-lexicographic generic initial ideal of the ambient Jacobian ideal has pure-second-variable cutoff 8 and a degree-7 minimal generator whose largest variable is the third. This explicitly contradicts the second-variable bound of Bigatti–Palezato–Torielli, also stated in OWR Report 5/2021. We prove the generic sectional dimensions on a nonempty open set of compatible flags, using scalar polar syzygies and Hilbert–Burch, and give exact rational checks. The arrangement is a classical parallel connection; the methods and the valid three-variable arrangement theorem are prior work. Historical novelty and independent discovery are not established. This unrefereed note was developed with extensive AI assistance and has not undergone conventional human peer review.

1 Statement and conventions

Let K be any field of characteristic zero. Use the order $x_1 > x_2 > x_3 > x_4$ for degree reverse lexicographic generic initial ideals. For an arrangement $\mathcal{A}$ with Jacobian ideal J , put

$$B = \text{gin}_{\text{revlex}}(J), \quad p = \min\{d \geq 0 : x_2^d \in B\}.$$

Conjecture 5.7 of Bigatti–Palezato–Torielli [1] asks that a minimal generator of B involving x_3 have degree at least p . Their notation is $d_0 = \min\{d : x_2^{d+1} \in B\}$, so $p = d_0 + 1$. The same second-variable statement appears as Conjecture 7 on printed page 234 of OWR Report 5/2021 [2], and as Conjecture 7.9 in [3, 4]. A first-variable exponent in an imported problem record was a transcription error; correcting it is source normalization, not a mathematical contribution.

Theorem 1. *In $S = K[x, y, z, w]$, let*

$$Q = xy(x - y)(x - 2y)zw(x + z + w), \quad J = (Q_x, Q_y, Q_z, Q_w).$$

For $B = \text{gin}_{\text{revlex}}(J) \subset K[x_1, x_2, x_3, x_4]$, one has $p = 8$, and B has a degree-7 minimal generator in $K[x_1, x_2, x_3]$ divisible by x_3 . In particular its largest variable is x_3 and its degree is strictly less than p .

The seven factors are distinct, and the coordinate factors make the central arrangement essential. Euler’s identity $xQ_x + yQ_y + zQ_z + wQ_w = 7Q$ shows that adjoining Q does not change J . The four polars have degree 6.

The arrangement itself is a classical construction: it is the parallel connection along the shared factor x of the rank-two four-element pencil $\{x, y, x - y, x - 2y\}$ and the rank-three four-element circuit $\{x, z, w, x + z + w\}$. Their matroids are $U_{2,4}$ and $U_{3,4}$. See Falk–Proudfoot [6, §4, Theorem 4.1] for the realization of parallel connections by identifying shared coordinates. No new-arrangement claim is made here.

Marchesi–Palezzato–Torielli prove the conjecture for arrangements in K^3 [5, Theorem 4.7]. Our ternary section retains *four ambient polars*. It is not the three-generated Jacobian ideal obtained by first restricting Q and then differentiating in three variables. Theorem 1 therefore respects that result.

2 Polar relations on an open set of sections

For an injective map $C : K^r \rightarrow K^4$, $r = 2, 3$, let R_r be its polynomial coordinate ring and write X, Y, Z, W for the four coordinate restrictions. Write $L_1, \dots, L_7$ for the restricted factors, $a_i \in K^4$ for the *ambient* coefficient row of the i th factor, and

$$f_j = Q_j \circ C \quad (j = x, y, z, w), \quad I_r = (f_x, f_y, f_z, f_w) \subset R_r.$$

Require that X, Y be independent and that all seven L_i be nonzero and pairwise nonproportional. For $r = 3$ also require that X, Z, W be independent. These are open rank conditions. Nonemptiness for compatible sections is proved below.

For a tuple $u = (u_x, u_y, u_z, u_w)$ of homogeneous linear forms in R_r , ambient differentiation gives

$$\sum_j u_j f_j = \sum_{i=1}^7 (a_i \cdot u) \prod_{h \neq i} L_h. \quad (1)$$

Reduce a zero relation modulo L_i . The quotient $R_r/(L_i)$ is a domain, and every other factor has nonzero image. Thus $L_i \mid a_i \cdot u$. Since both have degree one, there is a scalar λ_i with $a_i \cdot u = \lambda_i L_i$. Conversely, these proportionalities turn (1) into $(\sum_i \lambda_i) \prod_i L_i$. Consequently all linear syzygies are characterized by

$$a_i \cdot u = \lambda_i L_i \quad (1 \leq i \leq 7), \quad \sum_i \lambda_i = 0. \quad (2)$$

The same reduction for a constant tuple forces $a_i \cdot u = 0$ for every i . The coordinate-factor rows imply $u = 0$. Hence the four f_j are linearly independent and minimally generate I_r in degree 6.

This factor-divisibility calculation uses classical logarithmic-derivation ideas. Tohaneanu [7, §2.1] explains the linear case and explicitly traces its decomposition mechanism to Orlik–Terao [8, Proposition 4.29(3)]. We apply that elementary mechanism to restricted ambient polars; we do not assert a new general derivation theorem.

The first three factors $X, Y, X - Y$ force their scalars in (2) to coincide; $X - 2Y$ has the same scalar, say a . Call the remaining scalars b, c, d . The system becomes exactly

$$\begin{aligned} u_x &= aX, & u_y &= aY, & u_z &= bZ, & u_w &= cW, \\ (a - d)X + (b - d)Z + (c - d)W &= 0, & 4a + b + c + d &= 0. \end{aligned} \quad (3)$$

For $r = 3$, independence of X, Z, W gives $a = b = c = d$; the trace equation is $7a = 0$. Thus there is no linear syzygy. Writing $H_r(k) = \dim_K(R_r/I_r)_k$, we obtain

$$H_3(5) = 21, \quad H_3(6) = 28 - 4 = 24, \quad H_3(7) = 36 - 12 = 24. \quad (4)$$

For $r = 2$, the relation space of X, Z, W is one-dimensional. Choose $\rho_x X + \rho_z Z + \rho_w W = 0$ nonzero. All solutions of the vector equation in (3) are

$$a = d + t\rho_x, \quad b = d + t\rho_z, \quad c = d + t\rho_w, \quad 7d + t(4\rho_x + \rho_z + \rho_w) = 0.$$

The last equation has rank one, even if its coefficient of t vanishes. The map from its solution space to u is injective: $u = 0$ forces $a = b = c = 0$, and then $d(X + Z + W) = 0$ forces $d = 0$ and $t = 0$. There is therefore exactly one independent linear binary syzygy.

3 Binary Hilbert function and genericity

The ideal I_2 has height two. Indeed, work over an algebraic closure and suppose all four binary polars vanish at a projective point. Restricted Euler implies $\prod_i L_i = 0$. Exactly one L_i vanishes there, since nonproportional binary linear forms have distinct zeros. The ambient polar vector at that point is $a_i \prod_{h \neq i} L_h$, a nonzero vector, which is a contradiction. Thus there is no projective common zero, and R_2/I_2 is Artinian. This conclusion descends to K by extension of scalars.

The graded Hilbert–Burch theorem [9] gives a minimal resolution

$$0 \longrightarrow \bigoplus_{j=1}^3 R_2(-6 - e_j) \longrightarrow R_2(-6)^4 \longrightarrow I_2 \longrightarrow 0, \quad e_j \geq 1.$$

The finite-length quotient has Hilbert numerator $N(t) = 1 - 4t^6 + \sum_j t^{6+e_j}$, which has a double zero at 1. The equation $N'(1) = 0$ gives $\sum_j e_j = 6$. Exactly one linear syzygy means exactly one e_j is 1; the other two are at least 2. Thus the coefficient degrees are $\{1, 2, 3\}$ and

$$\text{HS}(R_2/I_2; t) = \frac{1 - 4t^6 + t^7 + t^8 + t^9}{(1 - t)^2}, \quad (H_2(6), H_2(7), H_2(8)) = (3, 1, 0). \quad (5)$$

For completeness the open conditions above hold on a compatible flag over every field in scope. Take

$$(x, y, z, w) = (s, t, u, s + 2t + 3u), \quad u = 2s + 3t. \quad (6)$$

In the ternary section, X, Y and X, Z, W have the required independence. The binary factor rows are

$$(1, 0), (0, 1), (1, -1), (1, -2), (2, 3), (7, 11), (10, 14).$$

They are nonzero with all pair determinants nonzero, as can be checked directly. Pairwise distinctness also holds before the binary restriction. A compatible full basis is the column basis of

$$C_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 7 & 11 & 3 & 1 \end{pmatrix}, \quad \det C_0 = 1.$$

Its first two columns give the binary section and its first three span the ternary section in (6).

The conditions therefore define a nonempty open set of compatible flags, whose lift to GL_4 is nonempty open. The calculations (4)–(5) hold throughout it. The variety GL_4 is irreducible, so this open set intersects the nonempty generic-initial open set of Galligo’s theorem [10]; see also [1, Theorem 4.4]. Since a characteristic-zero field is infinite, the required nonempty opens admit K -rational choices. Hence the displayed dimensions are the generic sectional dimensions over every such K . The explicit flag proves nonemptiness; favorable ranks at that flag alone would not prove genericity.

4 The generic-initial generator

The classical reverse-lexicographic section identity, for example Conca [11, Lemma 1.2] and [1, Theorem 6.2], identifies the generic dimensions with

$$H_r(k) = \dim_K(K[x_1, \dots, x_r]/B_r)_k, \quad B_r = B \cap K[x_1, \dots, x_r]. \quad (7)$$

In characteristic zero, B is strongly stable: if $x_j \mid m \in B$ and $i < j$, then $x_i m/x_j \in B$. Thus $x_2^k \in B$ implies that every binary degree- k monomial belongs to B . By (5), $x_2^7 \notin B$ and $x_2^8 \in B$. Ideal closure rules out any earlier pure power, so $p = 8$.

Put $A_3 = K[x_1, x_2, x_3]/B_3$. The cokernel of $x_3 : (A_3)_{k-1} \rightarrow (A_3)_k$ is $(K[x_1, x_2]/B_2)_k$. Hence

$$\dim_K \ker(x_3 : (A_3)_{k-1} \rightarrow (A_3)_k) = H_3(k-1) - H_3(k) + H_2(k). \quad (8)$$

This dimension is $21 - 24 + 3 = 0$ at $k = 6$ and $24 - 24 + 1 = 1$ at $k = 7$. There are no elements of B in degrees below 6, since taking an initial ideal preserves the graded Hilbert function. A degree-6 minimal generator of B_3 divisible by x_3 would, after division by x_3 , give a standard monomial in the first kernel. Thus no generator of B_3 involving x_3 has degree at most 6.

The second kernel is nonzero and is spanned by killed standard monomials. Choose a degree-6 standard monomial m with $x_3 m \in B_3$, and let v be a minimal generator dividing $x_3 m$. If $x_3 \nmid v$, then $v \mid m$, which contradicts standardness. Since v involves x_3 and has degree at most 7, the preceding paragraph forces $\deg v = 7$. It is also minimal in B , because a divisor of a monomial in the first three variables cannot involve x_4 . This proves Theorem 1.

Equation (8) is the classical sectional-matrix triangle calculation; compare [1, Theorem 6.6]. We use it inside B_3 and then prove minimality in B by divisibility. A row for the third truncation does not detect arbitrary full-ring generators involving later variables. No full generic initial ideal or particular exponent vector for v is needed.

5 Exact verification

The accompanying standard-library Python verification reconstructs Q over $\mathbb{Q}$, differentiates in four variables *before* restriction, and builds exact homogeneous multiplication matrices for (6). Their ranks are corroboration of the open-set proof:

Section variables	Total degree	Matrix shape	Rank	Quotient dimension
3	6	4×28	4	24
3	7	12×36	12	24
2	6	4×7	4	3
2	7	8×8	7	1
2	8	12×9	9	0

Rows are polynomial multiples of the four polars and columns are degree- k monomials. Exact nonzero minors and complete left-kernel bases certify both rank bounds. The binary syzygy is

$$11s f_x + 11t f_y - 73(2s + 3t) f_z + 25(7s + 11t) f_w = 0.$$

A constructive 4×3 Hilbert–Burch matrix with coefficient degrees $(1, 2, 3)$ is also certified by all its signed maximal minors. The files `reproduce_exact.py` and `EXACT_CERTIFICATE.json` reconstruct and verify these data; `orthogonal_checks.py` and `ORTHOGONAL_CERTIFICATE.json` add exact initial-ideal checks at specified rational coordinates and excluded-case controls. See the package README for commands requiring Python 3.11 or later. These finite coordinate checks make no additional genericity or full-gin claim.

6 Prior overlap, priority limits, and disclosure

An applicable prior family theorem deserves explicit credit. For this arrangement there is one codimension-two flat of multiplicity four, with three hyperplanes outside the pencil. Migliore–Nagel [12, §8.2, Theorem 8.11 and Corollary 8.12] give

$$0 \longrightarrow S(-7)^3 \oplus S(-9) \longrightarrow S(-6)^5 \longrightarrow J^{\text{top}} \longrightarrow 0,$$

where J^{top} is the intersection of the codimension-two primary components of J . This is substantive prior information about this family. It concerns a five-generated ideal, whereas J has four minimal ambient polars. Their §3 also stresses the distinction between restricting an ambient Jacobian ideal and taking the Jacobian ideal of a restricted arrangement. The proof above concerns the successive sections of J .

Historical priority remains unresolved. A bounded literature audit completed on 15 October 2026 did not identify an earlier complete predecessor in its inspected corpus. In particular, the full text of Ilardi’s 2018 article [13] was not obtained: publisher retrieval failed and institutional access was restricted. Its abstract does not exclude a relevant theorem or family. No assertion of firstness, independent discovery, or present openness follows from the accessible literature. The audit did not exhaust parameter families, related nine-factor generic initial ideals, or historical code, and it did not authenticate the candidate’s first public disclosure timestamp. The supported outcome here is an explicit counterexample with a checkable proof. We make no claim of smallest hyperplane count, a full generic initial ideal, or a positive-characteristic extension.

Extensive AI tools were used in solving, drafting, literature inspection, exact-verification development, and adversarial checking. The artifact has received multiple AI-assisted mathematical checks but no conventional human peer review and is unrefereed. Those checks are not evidence of historical priority or a substitute for such review. All cited construction and method credits apply regardless of the eventual priority determination.

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