

Counterexamples to Satoh's $\mathrm{SL}(2)$ representation-filtration conjecture

Alec Kriebel

Independent researcher

ORCID: [0009-0001-9320-500X](https://orcid.org/0009-0001-9320-500X)

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Abstract

Let $A_n(k)$ be the integral Andreadakis filtration of the automorphism group of a free group, and let $D_n(k)$ be the kernel of its action on J/J^{k+1} in the rational $\mathrm{SL}(2, \mathbb{C})$ matrix-entry algebra at the trivial representation. We construct elements of $D_n(5) \setminus A_n(5)$ for every $n \geq 4$, and of $D_n(6) \setminus A_n(6)$ for every $n \geq 3$. Thus the equality in all degrees conjectured in Satoh's 2018 report is false at every rank in its stated range. The construction uses a classical degree-five Lie identity, elementary free-group automorphisms, and integral Magnus coefficients. The proof works with actual powers of J , rather than only values along one-parameter representations.

1 The conjecture and the counterexamples

Write $F_n = \langle x_1, \dots, x_n \rangle$ and $\Gamma_1 F_n = F_n$, $\Gamma_{r+1} F_n = [\Gamma_r F_n, F_n]$. All lower central subgroups in this note are integral. Put

$$A_n(k) = \ker(\mathrm{Aut} F_n \longrightarrow \mathrm{Aut}(F_n/\Gamma_{k+1} F_n)).$$

For a representation $\rho : F_n \rightarrow \mathrm{SL}(2, \mathbb{C})$, let $a_{ij}(g)(\rho) = \rho(g)_{ij}$. Let R be the $\mathbb{Q}$ -algebra of functions generated by all $a_{ij}(g)$, and set

$$J = (a_{ij}(g) - \delta_{ij} \mid g \in F_n, 1 \leq i, j \leq 2) \subset R, \quad D_n(k) = \ker(\mathrm{Aut} F_n \longrightarrow \mathrm{Aut}(J/J^{k+1})).$$

We use the substitution action $a_{ij}(g)^\phi = a_{ij}(\phi(g))$. These are the component-function algebra and the full ideal quotient of [1, pp. 88–89], not a trace algebra or a single associated graded piece. That source records $A_n(k) \subseteq D_n(k)$ and equality for $1 \leq k \leq 4$, then conjectures equality for all $k \geq 1$ in the discussion with $n \geq 3$. See also [2, §4.2, Theorem 6.3] for the definitions and low-degree comparison.

Throughout, $[g, h] = ghg^{-1}h^{-1}$ for groups and $[U, V] = UV - VU$ for algebras. Define the words

$$w = [[[x_1, x_2], [x_1, x_3]], x_1], \tag{1}$$

$$v = [[[x_1, x_2], [x_1, [x_1, x_2]]], x_1]. \tag{2}$$

Theorem 1. *For every $n \geq 4$, the automorphism fixing all generators except $x_4 \mapsto x_4 w$ belongs to $D_n(5) \setminus A_n(5)$. For every $n \geq 3$, the automorphism fixing all generators except $x_3 \mapsto x_3 v$ belongs to $D_n(6) \setminus A_n(6)$.*

The degree-five identity used below is classical. With a change of sign and variables it is precisely Burde's identity (4); his Theorem 4.14 attributes its occurrence in a basis of $\mathfrak{sl}_2$ identities to Razmyslov [4]. We report that historical attribution through the inspected Burde source. The weighted substitution yielding the degree-six identity, Cayley–Hamilton, leading commutator expansions and Magnus expansions are also classical ingredients. No new polynomial identity or general PI method is claimed here. The contribution presented is their explicit application to the equality formulated in [1].

2 The matrix identity and actual ideal powers

Lemma 2. *For 2×2 matrices over any commutative ring,*

$$P(X, Y, Z) := [[[X, Y], [X, Z]], X] = 0.$$

Consequently $Q(X, Y) := P(X, Y, [X, Y]) = 0$.

Proof. Put $t = \text{tr } X$, $U = [X, Y]$ and $V = [X, Z]$. Cayley–Hamilton gives $X^2 - tX + \det(X)I = 0$, whence

$$XU + UX = [X^2, Y] = tU, \quad XV + VX = tV.$$

It follows that $XUV = (tU - UX)V = tUV - U(tV - VX) = UVX$. Likewise $XVU = VUX$, so X commutes with $UV - VU$. The second assertion is substitution. This elementary proof uses no division or diagonalization. $\square$

Lemma 3. *Let I be an ideal in a commutative ring S . If $1 + u$ and $1 + z$ are invertible matrices, with $u \in \text{Mat}_2(I^r)$ and $z \in \text{Mat}_2(I^s)$, $r, s \geq 1$, then*

$$[1 + u, 1 + z] - 1 = (uz - zu)(1 + u)^{-1}(1 + z)^{-1}.$$

Its entries lie in I^{r+s} , and its class modulo I^{r+s+1} is the bracket of the leading classes of u and z .

Proof. Subtract $(1 + z)(1 + u)$ from $(1 + u)(1 + z)$ and multiply on the right by the two displayed inverses. Both inverses are congruent to 1 modulo I , so their product does not change the class in degree $r + s$. $\square$

Let $M(g) = (a_{ij}(g)) \in \text{SL}_2(R)$ be the universal matrix of a word. Write $M(x_i) = 1 + B_i$, so $B_i \in \text{Mat}_2(J)$, and let X_i be its degree-one class in $\text{Mat}_2(\text{gr}_J R)$. Successive applications of Lemma 3 show that the degree-five class of $M(w) - 1$ is $P(X_1, X_2, X_3)$, and the degree-six class of $M(v) - 1$ is

$$[[[X_1, X_2], [X_1, [X_1, X_2]]], X_1] = P(X_1, X_2, [X_1, X_2]).$$

The latter substitution has weight two, giving total degree six. The associated graded ring is commutative, so Lemma 2 applies. The zero leading classes give the actual ideal containments

$$M(w) - 1 \in \text{Mat}_2(J^6), \quad M(v) - 1 \in \text{Mat}_2(J^7). \quad (3)$$

No assertion that a graded ring is a polynomial ring, no smoothness or reducedness assumption, and no passage from vanishing along arcs is needed.

3 Integral lower central obstructions

Let $T = \mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$ be the ring of noncommutative formal power series, filtered by total degree. The assignment $x_i \mapsto 1 + X_i$ defines a group homomorphism $E : F_n \rightarrow T^\times$, since $1 + X_i$ has its formal geometric inverse. The same commutator calculation, now in the degree filtration, gives

$$E(\Gamma_r F_n) \subseteq 1 + T_{\geq r} \quad (r \geq 1). \quad (4)$$

This follows inductively because $[1 + T_{\geq r}, 1 + T_{\geq s}] \subseteq 1 + T_{\geq r+s}$. Only this implication is required; neither injectivity of E nor a converse dimension-subgroup theorem is used.

The leading terms of $E(w) - 1$ and $E(v) - 1$ are the free associative polynomials $P(X_1, X_2, X_3)$ and $Q(X_1, X_2)$, respectively. They do not vanish in the free algebra. Indeed,

$$[X_1 X_1 X_2 X_1 X_3]P = -1, \quad [X_1 X_1 X_1 X_2 X_1 X_2]Q = 1, \quad (5)$$

where $[m]$ denotes the coefficient of the monomial m . For the first coefficient, in $[[U, V], X_1] = UVX_1 - VUX_1 - X_1UV + X_1VU$ with $U = [X_1, X_2]$, $V = [X_1, X_3]$, only $-X_1UV$ contributes to the indicated monomial. For the second take $U = X_1X_2 - X_2X_1$ and $V = X_1^2X_2 - 2X_1X_2X_1 + X_2X_1^2$; only X_1VU contributes, using the first monomial in each of V and U . Thus (4) proves

$$w \notin \Gamma_6 F_n \quad (n \geq 3), \quad v \notin \Gamma_7 F_n \quad (n \geq 2). \quad (6)$$

All coefficients are integral. Adjoining additional free generators leaves these coefficients unchanged, so these assertions require no stability or rational-isolator assumption.

4 Automorphisms and the full quotient action

Proof of Theorem 1. If a word u uses no x_t , the substitution $\phi_u(x_t) = x_t u$ fixing the other generators is an automorphism. Its inverse sends x_t to $x_t u^{-1}$ and fixes all others: both substitutions fix u , and their two compositions fix every free generator.

Apply this first to $(u, t) = (w, 4)$, and then to $(u, t) = (v, 3)$. For the moving generator,

$$M(\phi_u(x_t)) - M(x_t) = M(x_t)(M(u) - 1).$$

Equation (3) puts all its entries in J^{k+1} for $k = 5$ and 6 , respectively; every other generator matrix is fixed.

The entries of the finitely many matrices $M(x_i)$ generate R : products give the entries of products of words, and inverse entries are adjugate entries since all determinants are one. Therefore fixing these algebra generators modulo J^{k+1} fixes every element of R/J^{k+1} . Explicitly, the difference between a product of substituted generators and the original product is a sum of products each containing one generator difference. Each summand lies in J^{k+1} . The trivial representation is fixed by substitution, so J is invariant. Hence ϕ_u acts identically on the whole J/J^{k+1} , proving $\phi_u \in D_n(k)$.

Finally $x_t^{-1}\phi_u(x_t) = u$. The nonmembership statements (6) show that ϕ_u fails to act identically on $F_n/\Gamma_{k+1}F_n$. Thus $\phi_u \notin A_n(k)$, as required. $\square$

For each $n \geq 3$ the second construction already refutes equality in all degrees. The first gives the earlier failure in degree five when $n \geq 4$. We make no claim about degree five at rank three, failure in every degree, larger matrix sizes, character algebras, or the different comparison of $A_n(k)$ with the lower central series of IA_n .

5 Verification, related work and status

The accompanying archive preserves the original candidate packet and separate independent mathematical and priority audits. Exact computations check the matrix identities and finite ideal quotients. A distinct integral check uses upper unitriangular matrices. In size six, assigning

$$x_1 \mapsto I + E_{12} + E_{23} + E_{45}, \quad x_2 \mapsto I + E_{34}, \quad x_3 \mapsto I + E_{56}$$

sends w to $I - E_{16}$. In size seven, assigning

$$x_1 \mapsto I + E_{12} + E_{23} + E_{34} + E_{56}, \quad x_2 \mapsto I + E_{45} + E_{67}$$

sends v to $I + E_{17}$. Here E_{ij} is a matrix unit and other generators map to I . Direct integral multiplication verifies both identities. The d th lower central subgroup of the size- d unitriangular group is trivial, since commutators add distances above the diagonal. These give independent finite certificates for (6). The archive’s 489 colored-path evaluations recover all coefficients through the relevant degrees. These exact checks supplement the written proof; they are not a formal proof certificate or a literature certificate.

The bounded priority audit inspected the literal 2018 conjecture, Satoh’s public 2017 preprint version, and classical PI and related automorphism literature. It found no verified earlier complete application to this exact conjecture. Full journal texts of [3] and the 2021 publication related to [2], as well as some related 2025–2026 journal articles, were unavailable. Earlier versions, abstracts and related shorter sources are identified separately in the archive. No firstness, exhaustive novelty clearance, or continuously open status through 2026 is certified. The theorem is a negative answer to the equality as formulated in the inspected 2018 source.

AI assistance and review. AI tools were used extensively in developing the constructions, checking proofs and computations, investigating sources, and writing this note. This is an unrefereed preprint without independent human peer review. Internal adversarial AI reviews do not constitute human peer review. The authored note and supporting materials are released under CC BY 4.0; third-party full-text sources are not redistributed in the archive.

References

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