A Robinson counterexample to ideal-only tangency certificates

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Abstract

The classical Robinson sextic is nonnegative, but adding any nonnegative constant does not make it a sum of polynomial squares modulo the ideal of polynomials vanishing on its real tangency variety. This rules out both the exact radical certificate and the positive-constant certificate in the proposed unrestricted replacement of the gradient ideal. The same obstruction applies to the separate-cross-product ideal printed in the original report. Ten zero lines force the low-degree homogeneous parts of all square summands to vanish; an additional coordinate axis then makes the certificate impossible. The proof allows arbitrarily many summands of arbitrary degree and is entirely elementary. We credit the classical form and zero-set method, and make no claim of first bibliographic priority.

1 The proposed certificates

For $S = \mathbb{R}[x, y, z]$, let $\sum S^2$ denote finite sums of squares of elements of S. For $f \in S$, write f_x, f_y, f_z for its partial derivatives and set

$$J_f = (yf_x - xf_y,\ zf_x - xf_z,\ zf_y - yf_z), \qquad K_f = (yf_x, xf_y, zf_x, xf_z, zf_y, yf_z).$$

For an ideal $L \subseteq S$, define

$$V_{\mathbb{R}}(L) = \{u \in \mathbb{R}^3 : h(u) = 0 \text{ for every } h \in L\}, \qquad I_{\mathbb{R}}(L) = \{h \in S : h|_{V_{\mathbb{R}}(L)} = 0\}.$$

The ordinary radical is $\sqrt{L} = \{h : h^N \in L \text{ for some } N \geq 1\}$. In particular $L \subseteq \sqrt{L} \subseteq I_{\mathbb{R}}(L)$.

Schweighofer's Conjecture 2 [1, p. 811] proposes removing the minimum-attainment assumption from the gradient-ideal equivalence in Theorem 1 [1, p. 810] by replacing that ideal. The four conditions are global nonnegativity, nonnegativity on the real variety, membership in $\sum S^2 + \sqrt{L}$, and membership of $f + \varepsilon$ in $\sum S^2 + L$ for every $\varepsilon > 0$. The printed replacement displays separate products, giving K_f in three variables. The later catalogue formulation (UnsolvedMath record 30000717)[1] uses the tangency minors, giving J_f. We treat both explicit readings; we do not assert an author-approved correction of the printed formula. Removing an attainment assumption includes polynomials whose minimum *is* attained.

[1] https://huggingface.co/datasets/ulamai/UnsolvedMath.

Theorem 1. *Let*

$$R = x^6 + y^6 + z^6 - x^4y^2 - x^2y^4 - x^4z^2 - x^2z^4 \\ - y^4z^2 - y^2z^4 + 3x^2y^2z^2.$$

Then $R \geq 0$ on $\mathbb{R}^3$, but for $L = J_R$ or $L = K_R$ and every real $\varepsilon \geq 0$,

$$R + \varepsilon \notin \sum S^2 + I_{\mathbb{R}}(L).$$

Consequently $R \notin \sum S^2 + \sqrt{L}$, and $R + \varepsilon \notin \sum S^2 + L$ for every $\varepsilon > 0$.

Thus both proposed certificate implications from global nonnegativity fail. Robinson's form is classical; it already appears in the same report [1, p. 798, equation (2)]. Our purpose is its explicit application to these ideal-only certificates. The classical non-SOS property alone would not suffice, since an ideal remainder could change a representation.

2 The elementary zero-set obstruction

Put $a = x^2$, $b = y^2$, $c = z^2$ and write $R = F(a, b, c)$. Direct expansion gives

$$F = (a - b)^2(a + b - c) + c(a - c)(b - c).$$

The symmetric polynomial F may be evaluated after ordering $a \geq b \geq c \geq 0$; both displayed terms are then nonnegative. Hence $R \geq 0$, and its infimum is zero, attained at the origin.

Let Z consist of the ten vectors

$$(1, s, t), \quad (1, s, 0), \quad (1, 0, t), \quad (0, 1, t), \qquad s, t \in \{-1, 1\},$$

where each applicable sign varies independently. Substitution gives $R(v) = 0$ for every $v \in Z$, and homogeneity gives $R(uv) = 0$ for every real u.

Lemma 2. *A homogeneous cubic vanishing on Z is zero. The same holds in degrees zero, one and two.*

Proof. The six vectors with a zero coordinate force the cubic to have the form

$$p = Ax(x^2 - y^2 - z^2) + By(y^2 - x^2 - z^2) + Cz(z^2 - x^2 - y^2) + Dxyz.$$

At $(1, s, t)$ its value is $-A - Bs - Ct + Dst$. Summing these four values against the sign characters $1, s, t, st$ gives $-4A, -4B, -4C, 4D$, respectively. Thus all four coefficients vanish. If a homogeneous degree-d polynomial p_d, with $0 \leq d < 3$, vanishes on Z, then $x^{3-d}p_d$ is such a cubic. Since S is an integral domain, $p_d = 0$. □

The ten-zero cubic obstruction is classical; Reznick gives the same ten directions and cubic argument in [2, Section 4(b), p. 259]. We include the full argument, then use whole zero lines and an additional axis to address the ideal remainder and positive constant perturbations.

Every point of the lines $\mathbb{R}v$, $v \in Z$, is a global minimizer of R. Its gradient is therefore zero, and each line lies in both $V_{\mathbb{R}}(J_R)$ and $V_{\mathbb{R}}(K_R)$. Both varieties also contain the axis $(u, 0, 0)$: there $\nabla R = (6u^5, 0, 0)$, all cross-products vanish, and $R = u^6$.

Proof of Theorem 1. Fix $\varepsilon \geq 0$ and either ideal L. Suppose

$$R + \varepsilon = \sum_{j=1}^{m} q_j^2 + h, \qquad q_j \in S, \quad h \in I_{\mathbb{R}}(L).$$

On every zero line, h vanishes and $\varepsilon = \sum_j q_j(uv)^2$ for all real u. Hence each real univariate polynomial $q_j(uv)$ is bounded in absolute value by $\sqrt{\varepsilon}$ on $\mathbb{R}$, and must be constant. Write $q_j = \sum_d q_{j,d}$ in homogeneous parts. Comparing powers of u gives $q_{j,d}(v) = 0$ for every $d \geq 1$ and $v \in Z$. Lemma 2 yields $q_{j,1} = q_{j,2} = q_{j,3} = 0$.

On the coordinate axis the certificate gives

$$u^6 + \varepsilon = \sum_j q_j(u, 0, 0)^2.$$

Every nonzero restriction on the right has degree at most three. Indeed, if $D > 3$ is the largest such degree, the coefficient of u^{2D} is a nonempty sum of squares of nonzero real leading coefficients, which cannot cancel. But the degree-one, degree-two and degree-three homogeneous parts have all vanished identically. The restrictions must therefore be constant, contradicting the nonconstant left side. The inclusions preceding the theorem give its remaining conclusions. □

Constants were retained throughout: when $\varepsilon > 0$ the q_j need not vanish on the zero lines. The proof imposes no global degree bound on the summands, radicality hypothesis, or finiteness condition on the varieties.

3 Formulation boundaries

The following standard sphere-minimization argument also underlies the tangency-set infimum identity in Laurent's updated manuscript of 6 February 2010 [5, Section 7.2.3, Lemma 7.21, manuscript pp. 117–118]. For any polynomial f in n variables, nonnegativity on all of $\mathbb{R}^n$ is equivalent to nonnegativity on the real variety of its tangency minors. If $f(w) < 0$ and $w = 0$, the origin already belongs to that variety. Otherwise minimize f on the sphere of radius $\|w\|$. At a minimizer v, Lagrange multipliers give $\nabla f(v) = \lambda v$ and $f(v) \leq f(w) < 0$, so all tangency minors vanish. This proves the reverse implication without any assumption about the global infimum. For the Robinson example, the first two proposed conditions therefore hold and both certificates fail.

The literal product reading has a further, separate defect. We use a constant shift of the standard nonattaining example in [3, Section 1.6, equation (6)] and [1, p. 810]:

$$Q(x, y) = (1 - xy)^2 + y^2 - \tfrac{1}{2}, \qquad K_Q = (yQ_x, xQ_y).$$

Here $yQ_x = 2y^2(xy - 1)$ and $xQ_y = 2x(x(xy - 1) + y)$. The first equation forces $y = 0$ or $xy = 1$. The second then forces $x = 0$ in the former case, and equals 2, a contradiction, in the latter. Thus $V_{\mathbb{R}}(K_Q) = \{(0, 0)\}$. Nevertheless $Q(0, 0) = 1/2$ and $Q(2, 1/2) = -1/4$. Its infimum $-1/2$ is not attained: equality would require both $y = 0$ and $xy = 1$, whereas $Q(1/u, u) = u^2 - 1/2$ approaches the infimum. This diagnostic applies only to the product reading; it does not negate the tangency-minor equivalence just proved.

For the actual gradient ideal $G_R = (R_x, R_y, R_z)$, Euler's identity gives $6R = xR_x + yR_y + zR_z$, hence $R \in G_R$. There is no contradiction with the gradient-ideal theorem. Likewise the gradient-tentacle inequalities [3] and truncated tangency certificates [4, 6] use additional inequalities and SOS-multiplier terms; they are different certificates, which this note does not refute.

Verification, attribution and review status

The accompanying source archive preserves the original candidate and its historical audits, fresh independent reviews, and exact reproduction controls. The cubic evaluation determinant is -128 in the stated monomial order in the original proof. Symbolic checks supplement the universal proof; no numerical SDP failure or fixed SOS-degree cutoff is used as evidence of nonexistence. Robinson's form and the classical zero-set method retain their prior attribution. A bounded priority audit is provided separately; first priority and continued open status through 2026 are not established or asserted. Complete 2008 and 2009 Hà–Pham article texts were not obtained; comparisons with truncated certificates rely on later primary statements, and the access gaps remain in the audit.

AI tools were used extensively in developing, checking and presenting this argument, including independent AI-assisted adversarial reviews. This note is unrefereed and has not received independent human or journal peer review. Alec Kriebel is the author of the submitted note. The authored note and verification materials are released under CC BY 4.0; cited source documents retain their own terms and are not redistributed.

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