Robin half-space perturbations: a credited dimension-dependent answer to AIM Problem 1.4

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Abstract

We give a negative answer to the literal dimension-unrestricted wording of AIM Problem 1.4 on attractive Robin bound states under compact half-space perturbations. A nonzero smooth graph in $\mathbb{R}^4$, specified explicitly, has spectrum $[-1,\infty)$ at Robin strength one. In contrast, every connected, smooth, nontrivial compact half-space perturbation in ambient dimension two or three binds below the flat threshold at every positive strength, including overhangs and compact boundary components. Every compact Lipschitz graph in dimension at least four has no below-threshold spectrum at sufficiently weak strength. This is a credited exposition of standard mechanisms: Exner–Minakov already proved planar binding and used the exponential/Stokes normal-defect argument; the graph absence conclusion also follows from the established Hoang–Hundertmark–Richter–Vugalter bound after the explicit geometric comparison supplied here. We give both that implication and a self-contained Hardy–trace certificate. Historical priority of the exact remaining geometric formulations, novelty, worldwide open status and independent discovery remain unresolved. Extensive AI assistance was used; this note is unrefereed.

1 Question, conventions and attribution

Write $n = m+1 \geq 2$, $z = (x,y) \in \mathbb{R}^m \times \mathbb{R}$, and $H = \{y > 0\}$. The attractive Robin operator $L_{\beta,\Omega}$, $\beta > 0$, is the self-adjoint operator associated with

$$q_{\beta,\Omega}[u] = \int_\Omega |\nabla u|^2 \,\mathrm{d}z - \beta \int_{\partial\Omega} |u|^2 \,\mathrm{d}S, \qquad D(q) = H^1(\Omega). \tag{1}$$

Its outward condition is $\partial_\nu u = \beta u$; the flat threshold is $-\beta^2$. The negative parameter α on the AIM source page [1] equals $-\beta$. The relevant problem and its immediately preceding introduction impose no ambient-dimension restriction. Our negative answer is to that literal wording, without a claim about an unstated intended dimension or the question's current worldwide status.

A *smooth compact perturbation* means a connected open Ω with C^∞ boundary, $\Omega \triangle H$ bounded and $\Omega \neq H$. We allow disconnected boundary and overhangs. For the graph statement we use $f \in W^{1,\infty}_c(\mathbb{R}^m)$ and $\Omega_f = \{y > f(x)\}$.

Theorem 1. *The following statements hold.*

1. *Every smooth compact perturbation has $\sigma_{\mathrm{ess}}(L_{\beta,\Omega}) = [-\beta^2,\infty)$. In ambient dimensions $n = 2,3$, it has a discrete eigenvalue below $-\beta^2$ for every $\beta > 0$.*

2. *If $m \geq 3$ and $f \in W^{1,\infty}_c(\mathbb{R}^m)$, there is $\beta_0(f) > 0$ such that, for $0 < \beta \leq \beta_0(f)$,*

$$\sigma(L_{\beta,\Omega_f}) = \sigma_{\mathrm{ess}}(L_{\beta,\Omega_f}) = [-\beta^2, \infty).$$

3. *In $\mathbb{R}^4$, the nonzero C^∞ graph $f = h/100$, where*

$$h(x) = \begin{cases} \exp(-1/(1-|x|^2)), & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{2}$$

satisfies $\sigma(L_{1,\Omega_f}) = [-1, \infty)$. For every prescribed $\beta_* > 0$, a sufficiently small nonzero $f = \varepsilon h$ gives another absence example at β_*.

The planar single-infinite-C^4-curve, same-line-tail case of part 1 is due to Exner–Minakov [2, Theorem 5.2]. Their proof already uses $\psi(x)e^{-\beta y}$ and Stokes' formula; its oriented curve factor is exactly the planar specialization of our $-(1+\nu_y)$ defect. We credit the mechanism as well as the theorem. The argument below verifies the additional boundary components and dimension-three scope directly. These scopes' exact historical priority remains unknown. Part 2 and the explicit example are also corollaries of the published massive-relativistic bound [3, Theorem A.6], through the all-form comparison in Section 5. We do not present the dimensional threshold or the flat Robin/relativistic reduction as new mechanisms. The latter is established in Frank–Laptev [4]; ground-state null sequences and criticality are established frameworks [5].

The theorem does not assert absence for one fixed nonzero graph at all couplings, classify arbitrary nongraph perturbations in $n \geq 4$, or exclude embedded eigenvalues. Equality of spectral sets with the essential interval alone does not exclude embedded point spectrum.

2 Forms and essential spectrum

Finite boundary charts on the altered compact part, together with the flat trace estimate elsewhere, give for every $\epsilon > 0$

$$\|\operatorname{Tr} u\|^2_{L^2(\partial\Omega)} \leq \epsilon \|\nabla u\|^2 + C_\epsilon \|u\|^2.$$

Thus (1) is semibounded and closed, with a sufficiently shifted form norm equivalent to H^1. Smooth functions compactly supported on the closure are a form core in the smooth case. For Lipschitz graphs the bi-Lipschitz flattening $(x,t) \mapsto (x, t + f(x))$ transfers density from H. Trial functions need not satisfy the operator's Robin boundary condition. Integration on flat vertical fibers gives

$$Q_H[u] := q_{\beta,H}[u] + \beta^2\|u\|^2 = \|\nabla_x u\|^2 + \|\partial_y u + \beta u\|^2 \geq 0. \tag{3}$$

For any geometry in the theorem choose a smooth real partition $\chi^2 + \zeta^2 = 1$, with χ compactly supported and ζ zero near all geometric changes. The function ζu, extended across the missing part where necessary, lies in $H^1(H)$. IMS localization and semiboundedness of the compact part imply, for a fixed bounded K,

$$Q_\Omega[u] := q_{\beta,\Omega}[u] + \beta^2\|u\|^2 \geq -C \int_K |u|^2. \tag{4}$$

Indeed the exterior form is nonnegative by (3); the compact form and IMS error are bounded below by local L^2 mass. If $\lambda < -\beta^2$ were essential, a normalized weakly null operator Weyl sequence at λ would have bounded H^1 norm by trace coercivity. Local Rellich compactness makes its mass in K tend to zero, contradicting (4) and $Q[u_j] \to \lambda + \beta^2 < 0$.

Conversely let $\eta \in C_c^\infty(\mathbb{R}^m)$ have norm one and choose

$$\eta_R(x) = R^{-m/2}\eta((x - a_R)/R), \qquad u_R(x, y) = \sqrt{2\beta}\,\eta_R(x)e^{ik\cdot x}e^{-\beta y}.$$

Choose a_R so the entire tangential support escapes to infinity and lies outside the projection of every alteration. Extend by zero off those flat fibers. Integration by parts against all H^1 tests shows u_R is in the operator domain: it has the flat Robin condition on its boundary support and $Lu_R = -\Delta u_R$. This uses no global H^2 domain characterization. Its norm is one, it tends weakly to zero, and its residual at $|k|^2 - \beta^2$ has norm $O(|k|R^{-1}) + O(R^{-2})$. These operator Weyl sequences prove

$$\sigma_{\text{ess}}(L_{\beta,\Omega}) = [-\beta^2, \infty). \tag{5}$$

Below this interval spectrum is discrete, of finite multiplicity, and can accumulate only at the threshold.

3 Normal defect and low-dimensional binding

Put $\phi = e^{-\beta y}$. Since $\Delta\phi = \beta^2\phi$ and $\partial_\nu\phi = -\beta\nu_y\phi$, integration by parts gives, first for compactly supported core functions v,

$$Q_\Omega[\phi v] = \int_\Omega \phi^2|\nabla v|^2 - \beta\int_{\partial\Omega}(1 + \nu_y)\phi^2|v|^2\,\mathrm{d}S. \tag{6}$$

Real parts of the cross term give the same formula for complex v. The integrand in

$$B_\beta(\Omega) = \int_{\partial\Omega}(1 + \nu_y)e^{-2\beta y}\,\mathrm{d}S$$

is nonnegative and supported on the compact altered part. It is finite and strictly positive. To see the latter, zero total would force $\nu = -e_y$ everywhere by continuity. Each boundary component would then be locally horizontal, open in its containing horizontal plane, and closed in $\mathbb{R}^n$, hence the full plane. Compact agreement permits only $y = 0$. Connectedness and agreement with H at infinity then force $\Omega = H$, a contradiction. Overhangs and compact obstacle boundaries do not change this sign or this argument.

Take R_0 larger than the projection of all alterations and a compactly supported Lipschitz $\psi(x)$ equal to one near B_{R_0}. The function $\phi\psi$ lies in $H^1(\Omega)$: the domain is bounded below, and its upward tail is exponentially integrable. Vertical truncations above the alteration have vanishing exponential errors, so (6) applies. The support of $\nabla\psi$ sees exactly flat whole vertical fibers, whence

$$Q_\Omega[\phi\psi] = \frac{1}{2\beta}\int_{\mathbb{R}^m}|\nabla\psi|^2 - \beta B_\beta(\Omega). \tag{7}$$

For $m = 1$ use two linear tapers of length L outside $[-R_0, R_0]$; their total energy is $2/L$. For $m = 2$ use the radial logarithmic taper from R_0 to R, of energy $2\pi/\log(R/R_0)$. Both tend to zero. Equation (7) is negative for a sufficiently large cutoff; (5) and min–max prove part 1.

This is an actual-domain null-sequence argument. In the terminology of [5], the auxiliary pair $P_0 = -\Delta + \beta^2$, $B_0u = \partial_\nu u + \beta\nu_y u$ has the positive solution ϕ and transformed energy $\int_\Omega \phi^2|\nabla v|^2$. The same tangential cutoffs, with vertical heights chosen afterwards to kill each tail error and normalization on a fixed interior ball, supply its null sequence. Smooth approximation supplies the required local core. Subtracting the strictly positive compact boundary defect gives binding. Thus we verify geometry and trace mass rather than assume flat criticality transfers to an altered domain. The interior-potential perturbation remark in [5, Remark 5.22] is not a theorem about arbitrary domain changes.

4 An elementary all-form graph estimate

Let $m \geq 3$, $K = \operatorname{supp} f \subset \overline{B}_R$, and put

$$M = \|f\|_\infty, \quad G = \|\nabla f\|_\infty, \quad D = \sqrt{1+G^2} - 1, \quad A = \frac{4R^2}{(m-2)^2}, \quad c = \frac{1}{2(1+G^2)}.$$

Classical tangential Hardy gives

$$\int_K |a|^2 \leq A \int_{\mathbb{R}^m} |\nabla a|^2. \tag{8}$$

For completeness, expand $0 \leq \int |\nabla a + a_0 p_\delta a|^2$ with $a_0 = (m-2)/2$ and $p_\delta = x/(|x|^2+\delta^2)$. Since $\operatorname{div} p_\delta = ((m-2)|x|^2 + m\delta^2)/(|x|^2+\delta^2)^2$, integration by parts yields

$$\int |\nabla a|^2 \geq a_0^2 \int \frac{|x|^2}{(|x|^2+\delta^2)^2} |a|^2.$$

The omitted $a_0 m\delta^2$ term is nonnegative for $m \geq 3$. Fatou and density prove Hardy; $1 \leq R^2/|x|^2$ on the ball gives (8).

For a compact-support flattened core write $u(x, t+f(x)) = e^{-\beta(t+f(x))} w(x,t)$, $t > 0$, and set

$$E_x = \int_H e^{-2\beta t} |\nabla_x w|^2, \quad E_t = \int_H e^{-2\beta t} |w_t|^2, \quad E = E_x + E_t.$$

If $L_K = \int_{K\times(0,\infty)} e^{-2\beta t}|w|^2$ and $T_K = \int_K |w(x,0)|^2$, then $L_K \leq A E_x$ and vertical integration gives

$$T_K = 2\beta L_K - 2\operatorname{Re} \int_{K\times(0,\infty)} e^{-2\beta t} \overline{w} w_t \leq (2\beta A + \sqrt{A}) E. \tag{9}$$

The last step uses $2\sqrt{E_x E_t} \leq E_x + E_t$. The weak chain rule and vertical-fiber integration give the exact identity

$$\begin{aligned} Q_f[u] = &\int_H e^{-2\beta(t+f)} (|\nabla_x w - \nabla f\, w_t|^2 + |w_t|^2) \\ &- \beta \int_{\mathbb{R}^m} e^{-2\beta f} (\sqrt{1+|\nabla f|^2} - 1) |w(x,0)|^2 \, \mathrm{d}x. \end{aligned} \tag{10}$$

In particular the Robin surface Jacobian is retained; the fiber endpoint $+\beta|w(0)|^2$ combines with $-\beta\sqrt{1+|\nabla f|^2}|w(0)|^2$. For $|g| \leq G$, the triangle inequality squared gives

$$|a|^2 + |b|^2 \leq 2|a - gb|^2 + (2G^2+1)|b|^2 \leq 2(1+G^2)(|a-gb|^2 + |b|^2).$$

Equations (9)–(10) therefore imply

$$Q_f[u] \geq [e^{-2\beta M} c - \beta e^{2\beta M} D(2\beta A + \sqrt{A})] E. \tag{11}$$

Consequently

$$\beta D e^{4\beta M} (2\beta A + \sqrt{A}) \leq c \tag{12}$$

implies $Q_f \geq 0$ on all of $H^1(\Omega_f)$. Indeed we first used a dense compact-support core, where the exponential and its inverse are bounded, then use form continuity and H^1 density. We never assume multiplication by the globally growing $e^{\beta y}$ preserves H^1. For fixed f, the left side of (12) tends to zero as $\beta \downarrow 0$; an interval $(0, \beta_0]$ follows. Together with (5) this proves part 2. The flat case $f = 0$ holds for all strengths. The homogeneous estimate (8) is unavailable in $m = 1, 2$, consistently with their capacity cutoffs.

5 A complete earlier-theorem implication

We now give the alternative derivation from the published Hoang–Hundertmark–Richter–Vugalter theorem [3], to make the attribution substantive. Set $U(x,t) = e^{\beta f(x)}u(x,t+f(x))$. This is a bounded H^1 isomorphism because flattening is bi-Lipschitz and $e^{\beta f}$ and its inverse are bounded Lipschitz multipliers. Writing $b = U_t + \beta U$, direct expansion gives, on all of H^1 by density,

$$\begin{aligned} Q_f[u] = &\int_H e^{-2\beta f}(|\nabla_x U - \nabla f\, b|^2 + |b|^2) \\ &- \beta \int e^{-2\beta f}(\sqrt{1+|\nabla f|^2} - 1)|U(x,0)|^2. \end{aligned}$$

Hence, with $a_\beta = e^{-2\beta M}c > 0$, $v_\beta = \beta D e^{4\beta M}/c$ and $E_\beta[U] = \int_H (|\nabla_x U|^2 + |U_t + \beta U|^2)$,

$$Q_f[u] \geq a_\beta \left(E_\beta[U] - v_\beta \int_K |U(x,0)|^2 \right). \tag{13}$$

For the unitary tangential Fourier transform put $\rho = (|p|^2 + \beta^2)^{1/2}$ and $a_0 = U(\cdot,0)$. Completion of the vertical square gives

$$\begin{aligned} E_\beta[U] &= \int_{\mathbb{R}^m} \int_0^\infty |\partial_t \widehat{U} + \rho \widehat{U}|^2 \,\mathrm{d}t\,\mathrm{d}p + \int_{\mathbb{R}^m} (\rho - \beta)|\widehat{a}_0|^2 \,\mathrm{d}p \\ &\geq \langle a_0, T_\beta a_0 \rangle, \qquad T_\beta = \sqrt{-\Delta + \beta^2} - \beta. \end{aligned} \tag{14}$$

The identity extends from the core; $a_0 \in H^{1/2}$. Frank–Laptev [4, Theorem 3.2 and Lemma 3.4] already establish the flat Robin/relativistic duality, but their form-compact boundary-potential hypothesis does not include the nondecaying attractive constant baseline. We use (14) directly rather than apply that theorem outside its hypotheses.

For fixed $\beta > 0$ the form norm of $T_\beta + 1$ is equivalent to $H^{1/2}(\mathbb{R}^m)$. The bounded potential $V = -v_\beta \mathbf{1}_K$ is relatively form small. Multiplication by $|V|^{1/2}$ is compact from $H^{1/2}$ to L^2, by local Rellich compactness on a ball containing K. Thus the essential spectrum stays $[0,\infty)$ and the hypotheses for the quantitative bound in [3] are satisfied.

For $m \geq 3$, published Theorem A.6 gives

$$N(T_\beta + V) \leq \frac{\theta^2}{(1-2\theta)^2} C_m \int (\theta^{-2} V_-)^{m/2} (\theta^{-2} V_- + 2\beta)^{m/2}, \tag{15}$$

where $0 < \theta < 1/2$, $C_m = \frac{m}{m-2}\frac{\omega_m}{(2\pi)^m}$, ω_m is the unit-ball volume, and N counts strictly negative eigenvalues with multiplicity. The version of record was published in 2022 and appears in the 2023 journal volume. Its source mass is our β and its dimension is our boundary dimension m. The 2017 arXiv v2 Theorem 3.6 instead has denominator $(1-4\theta)^2$ and $0 < \theta < 1/4$. Our choice $\theta = 1/4$ uses only the authenticated *published* formula; it must not be inserted in that preprint formula.

Take $\theta = 1/4$. For $0 < \beta \leq 1$ set $C_f = De^{4M}/c$, so $v_\beta \leq \beta C_f$. Then

$$N(T_\beta - v_\beta \mathbf{1}_K) \leq L_f \beta^m, \quad L_f = \frac{|K| C_m}{4} [16 C_f (16 C_f + 2)]^{m/2}. \tag{16}$$

For $D > 0$ choose $\beta_0 = \min\{1, (2L_f)^{-1/m}\}$; for $D = 0$, compact support makes $f = 0$. Since N is an integer, (16) gives $N = 0$. There is no other negative spectrum, by the verified essential-spectrum hypothesis; hence $T_\beta - v_\beta \mathbf{1}_K \geq 0$. Equations (13)–(14) prove $Q_f \geq 0$ on its entire form domain. This is a complete earlier-theorem corollary for part 2, not an effective-operator heuristic. Its exact geometric historical priority remains unknown.

6 Two exact certificates for the specified graph

The flat extension of $e^{-1/s}$ at $s = 0$ makes (2) smooth. It is positive at zero. For $r = |x| < 1$ and $q = (1 - r^2)^{-1} \geq 1$,

$$|\nabla h| = 2rq^2e^{-q} \leq 2q^2e^{-q} \leq 8/e^2 < 2.$$

Thus for $f = h/100$, $M < 1/100$, $G < 1/50$, $D < 1/5000$, and $c > 1250/2501$. At $\beta = 1$ and $m \geq 3$, $A \leq 4$ and $2A + \sqrt{A} \leq 10$. The power-series comparison $e^{1/25} < 1/(1 - 1/25) = 25/24$ gives

$$De^{4M}(2A + \sqrt{A}) < 1/480 < 1250/2501 < c.$$

The rational margin is $1250/2501 - 1/480 = 597499/1200480 > 0$. This proves (12) exactly, without numerical spectral input.

Independently, for $m = 3$, (15) gives a certificate from the prior bound. Here $v_1 < 2501/6000000$; putting $s = 16v_1$ gives $s < 2501/375000 < 1/100$ and $s(s + 2) < 201/10000 < 1/40$. Since $C_3 = 1/(2\pi^2)$ and $|K| \leq 4\pi/3$,

$$N(T_1 - v_1\mathbf{1}_K) \leq \frac{[s(s + 2)]^{3/2}}{6\pi} < 1/240 < 1.$$

We used $s(s + 2) < 1$ and $\pi > 1$ in the final coarse strict bound. Both routes establish part 3 with (5). The graph and its domain are connected and smoothly diffeomorphic to $\mathbb{R}^3$ and H, respectively; $\Omega_f \triangle H$ lies in a bounded unit cylinder. For each fixed prescribed β_*, replacing f by εh makes $M, G \to 0$ and $D = O(\varepsilon^2)$, so either certificate eventually applies. The graph is allowed to depend on β_*.

7 Source and verification limits

The standard large-strength curvature binding theorem of Pankrashkin–Popoff [6] is compatible with weak-strength absence. The 2017 delta-interaction work and slides [7] describe Robin expectations, not the Robin theorems proved here. Deformation amplitude and Robin strength are distinct parameters. Delta absence does not imply selected-side Robin absence: the two-sided Robin form has the larger direct sum form domain and its restriction to matching traces is the delta form at strength 2β. Its form order points the wrong way for that inference. The full printed magnetic model of [8] requires $B > 0$ comparable to β^σ, $\sigma \geq 2$, and sufficiently large β; it does not apply to our zero-field statements.

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AI tools were used extensively in solving, checking, source comparison and writing. This independent research note is unrefereed, has not received human peer review, and is not a proof-assistant certificate. Its finite checks verify selected exact algebra and custody; they do not replace the universal arguments above. The accompanying edition records the required subsequent whole-package adversarial review gate separately from historical math audits. The paper and newly authored support materials are offered under CC BY 4.0.

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