

A smooth Reshetnyak-harmonic sphere with unbounded distortion

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Abstract

We construct a compact smooth Riemannian sphere and a smooth homeomorphism from the round sphere which locally minimizes Reshetnyak energy against every same-trace Sobolev competitor, but has essentially unbounded infinitesimal distortion. This gives a negative answer for the Reshetnyak member of the general-energy harmonic-sphere question posed in the inspected 2025 texts of Meier–Vikman–Wenger. Scalar calibration on a flat belt and an exact two-form calibration on conformal caps prove the full local comparison. The map does not minimize energy in its homotopy class and is not Korevaar–Schoen harmonic. Separately, an angular-stress argument gives the nonsharp bound $4 + \sqrt{17}$ for locally Korevaar–Schoen-minimizing spheres into complete metric spaces; the sharper classical $\mathrm{SL}(2)$ -minimal bound also applies. Classical ingredients are credited; first priority is not claimed.

1 Question, energies, and exact conclusions

Question 1 of Meier’s Oberwolfach contribution [2, p. 1625], also Question 1.9 of Meier–Vikman–Wenger [1], asks whether harmonic spheres are infinitesimally quasiconformal in its stated metric setting. The energy is fixed from a class including Reshetnyak and Korevaar–Schoen energies [1, Definition 3.1]. We use the precise neighborhoodwise definition: each source point has an open neighborhood U such that, for every Lipschitz domain $\Omega \Subset U$ and every $v \in W^{1,2}(\Omega, X)$ with the trace of $u|_\Omega$, the chosen energy of $u|_\Omega$ is at most that of v [1, Definition 6.1]. Competitors may leave the reference image and need not be continuous or homotopic to the reference map.

For a smooth Riemannian target, the energies in conformal source coordinates are

$$E_+^2(v; \Omega) = \int_\Omega \|dv\|_{\mathrm{op}}^2, \quad E_{\mathrm{KS}}(v; \Omega) = \int_\Omega (\sigma_1(dv)^2 + \sigma_2(dv)^2).$$

The conformal source factor cancels. For a metric differential seminorm s , let $M(s)$ and $m(s)$ be its maximum and minimum on the Euclidean unit circle. Infinitesimal quasiconformality with constant Q means $M(s) \leq Qm(s)$ almost everywhere; the zero seminorm satisfies this inequality.

Theorem 1. *There is a compact smooth Riemannian surface X , diffeomorphic to S^2 , and a smooth homeomorphism $u : S^2 \rightarrow X$ such that u is locally minimizing for E_+^2 in the preceding full Sobolev sense, while no finite Q bounds its infinitesimal distortion. The target is geodesic, admits a local quadratic isoperimetric inequality, and every sufficiently small-diameter continuous sphere in X is null-homotopic.*

Theorem 2. *Let X be complete and $u \in W^{1,2}(S^2, X)$ be locally minimizing for Korevaar–Schoen energy in the standard metric-Sobolev framework of [1]. Then u is infinitesimally $(4 + \sqrt{17})$ -quasiconformal. No target curvature, topology, or isoperimetric hypothesis is needed.*

Theorem 1 answers the inspected general-energy question negatively. Theorem 2 addresses a separate fixed-energy restriction. Energy comparability does not identify their exact minimizers. The source binding is to the byte-checked arXiv v1 and the Oberwolfach report. The revised 2026 journal article exists, but its mathematical text was not accessible to this review; unchanged final wording or openness through 2026 is not asserted.

2 Two local comparisons with unrestricted competitors

Lemma 3. *Suppose a target patch is Euclidean with coordinates (s, y) , and a smaller set V has y 1-Lipschitz for the whole target distance. In source coordinates (t, x) , a map $u(t, x) = (h(t), x)$ into V with $|h'| \leq 1$ minimizes E_+^2 on every relatively compact Lipschitz subdomain against all same-trace Sobolev competitors in the whole target.*

Proof. The McShane extension [6]

$$F(q) = \inf_{p \in V} \{y(p) + d_X(p, q)\}$$

is globally 1-Lipschitz and agrees with y on V ; it is finite in the compact target used below. For any competitor v , scalar Sobolev composition gives $f = F \circ v \in W^{1,2}(\Omega)$, with trace x and $\int_\Omega |\nabla f|^2 \leq E_+^2(v; \Omega)$. Put $w = f - x \in W_0^{1,2}(\Omega)$. Integration of its weak x -derivative is zero, by compactly supported approximation. Therefore

$$E_+^2(v; \Omega) \geq \int_\Omega |\nabla f|^2 = |\Omega| + \int_\Omega |\nabla w|^2 \geq |\Omega| = E_+^2(u; \Omega).$$

There is no restriction on the competitor’s image. In a smooth target, a sufficiently small strongly geodesically convex ball inside a flat coordinate patch supplies the ambient-distance premise: its minimizing geodesics remain in the patch, so coordinate differences are bounded by their lengths. $\square$

Lemma 4. *A smooth orientation-preserving conformal map into an oriented compact smooth Riemannian sphere locally minimizes E_+^2 against every same-trace $W^{1,2}$ competitor.*

Proof. Write $A = \text{Area}(X)$. Shrink a source neighborhood U until its compact image supports a smooth cutoff $0 \leq \eta \leq 1$ which equals one near that image and has $0 < a := \int_X \eta dA < A/2$. Set

$$b = \frac{a}{A - a} < 1, \quad \omega = ((1 + b)\eta - b) dA.$$

This two-form has comass at most one, agrees with the area form near $u(U)$, and integrates to zero on X . Top-degree de Rham cohomology gives a global smooth one-form α with $d\alpha = \omega$. The standard calibration comparison [7] gives, pointwise for every competitor,

$$v^*\omega \leq |\text{Jac } v| dA_{\text{source}} \leq \|dv\|_{\text{op}}^2 dA_{\text{source}};$$

both inequalities are equalities for the reference conformal map. We justify the needed identity $\int_\Omega v^*\omega = \int_\Omega u^*\omega$ at the critical Sobolev regularity, rather than assuming smooth target-valued density.

A finite atlas and partition of unity write $\alpha = \sum_j f_j dg_j$ with globally smooth real functions on X . Coordinate functions are extended with cutoffs equal to one near the corresponding coefficient supports. For bounded scalar $F, G \in W_{\text{loc}}^{1,2}$, mollification proves $d(F dG) = dF \wedge dG$ in distributions: uniformly bounded approximants for F give L^2 convergence of the one-forms by dominated convergence and strong L^2 gradient convergence; products of the gradients converge in L^1 . Applying scalar composition and the chain rule thus gives $d(v^* \alpha) = v^* \omega$.

Matching traces glue v on Ω to u on $U \setminus \Omega$ as a Sobolev map $\tilde{v}$. This can also be checked in a smooth embedding of the compact target. The L^2 one-form $\beta = \tilde{v}^* \alpha - u^* \alpha$ is supported in $\bar{\Omega} \Subset U$. Pair its distributional derivative with a smooth cutoff equal to one there. The cutoff derivative misses its support, so

$$0 = \int_U d\beta = \int_{\Omega} (v^* \omega - u^* \omega),$$

where the middle integral denotes that cutoff pairing. Integrating the calibration inequalities proves the assertion. Extra bubbles, orientation reversal, and target excursions do not alter this exact-form identity. $\square$

3 The smooth compact counterexample

Use the cylinder $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ with coordinates (s, θ) and add its two ends as sphere poles. Define

$$B(r) = \begin{cases} e^{-1/r} & r > 0, \\ 0 & r \leq 0, \end{cases} \quad H(r) = \frac{B(r)}{B(r) + B(1-r)}, \quad \rho(s) = e^{-|s|H(|s|-2)}, \quad g_X = \rho(s)^2(ds^2 + d\theta^2).$$

The denominator never vanishes. The metric is flat for $|s| \leq 2$. At the negative end $z = e^{s+i\theta}$ makes it exactly $|dz|^2$ for $s \leq -3$; at the positive end $\zeta = e^{-s-i\theta}$ does the same for $s \geq 3$. It therefore extends smoothly and positively over both poles. The apparent absolute value is harmless since its coefficient vanishes near zero.

Compact smooth Riemannian geometry gives completeness and geodesicity, and a uniform contractible normal-ball radius. Every sufficiently small-diameter sphere image is contained in such a ball. To verify local quadratic isoperimetry, choose uniformly controlled nested charts with convex Euclidean images. A sufficiently short Lipschitz loop lies in a smaller chart. Its Euclidean image of length at most $L\ell$ can be coned to one of its points within distance $L\ell/2$, giving cone area at most $L^2\ell^2/4$. Returning through the inverse chart gives area at most $L^4\ell^2/4$. The radial cone is Sobolev and retains the prescribed Lipschitz boundary parametrization. Compactness makes the scale and constants uniform.

On the round source sphere use cylinder coordinates (t, θ) , with metric $g_0 = \text{sech}^2 t (dt^2 + d\theta^2)$. Put

$$\eta_0(t) = 1 - H((16t^2 - 1)/3), \quad a(t) = \eta_0(t)t^2 + 1 - \eta_0(t), \quad h(t) = \int_0^t a(r) dr.$$

Here a is smooth and even, $0 \leq a \leq 1$, vanishes only at zero, equals t^2 for $|t| \leq 1/4$, and equals one for $|t| \geq 1/2$. Thus h is a smooth, strictly increasing odd homeomorphism of $\mathbb{R}$, with

$$\begin{aligned} h(t) &= t^3/3 \quad (|t| \leq 1/4), \\ h(t) &= t - c \quad (t \geq 1/2), \quad h(t) = t + c \quad (t \leq -1/2), \\ c &= \int_0^{1/2} (1 - a(r)) dr \in (0, 1/2). \end{aligned}$$

Define $u(t, \theta) = (h(t), \theta)$ and send the poles to the corresponding target poles. Its pole expressions are multiplication by e^c in the respective complex coordinates, so it is a globally smooth homeomorphism. The inverse need not be smooth at the equator; that is not a hypothesis. Smoothness on the compact source implies $u \in W^{1,2}(S^2, X)$.

For $|t| < 1$, the image is in the flat target belt since $|h(t)| \leq |t|$. Around each such point choose a smaller source neighborhood mapping into a strongly convex target ball, and lift the angular coordinate to a real x . Lemma 3 proves every required fixed-trace comparison there. For $|t| > 1/2$ and at the poles, the map is orientation-preserving conformal; Lemma 4 proves every required comparison. These open regimes overlap and cover the sphere, including both seam circles.

For $0 < |t| < 1/4$, the singular values before the common source conformal rescaling are 1 and t^2 . Their ratio is t^{-2} . Given finite $Q \geq 1$, choose $0 < \delta < \min\{1/4, 1/\sqrt{Q}\}$. The band $0 < |t| < \delta$ has positive round area $4\pi \tanh \delta$ and ratio greater than Q . This proves essential unboundedness and Theorem 1.

For completeness, the maximum-square seminorm energy is admissible: continuity, monotonicity, degree-two homogeneity, and rotation invariance are immediate. Its unit sublevel consists of uniformly bounded 1-Lipschitz seminorms on the unit circle, compact by Arzelà–Ascoli. Quasiconvexity follows from convexity of the squared operator norm: a smooth fixed-affine-boundary map into a normed space has average derivative equal to that affine derivative, and Jensen’s inequality gives the energy comparison. No immersion restriction is needed.

Two distinctions that prevent false contradictions

The conformal coordinate identity $f(t, \theta) = (t, \theta)$ is homotopic to u and has $E_+^2(f) = \text{Area}(X)$. Changing variables $s = h(t)$ gives

$$E_+^2(u) - \text{Area}(X) = 2\pi \int_{-1/2}^{1/2} (1 - a(t)) dt = 4\pi c > 0.$$

Thus u is not a global homotopy minimizer. In a flat rectangle with $0 < t < 1/4$, perturb its first coordinate by $\varepsilon\psi$, where $\psi \geq 0$ is smooth, nonzero, and compactly supported. The KS first variation is

$$2 \int h' \partial_t \psi = -2 \int h'' \psi = -4 \int t \psi < 0.$$

Small perturbations keep the image flat and retain the trace, so u is not KS-harmonic. Nonsmooth operator-norm energy does not supply the unique linear stress used next; local supporting stresses cannot simply be patched globally.

4 The separate Korevaar–Schoen theorem

The metric-Sobolev approximate differential and domain-diffeomorphism chain rule are standard inputs [4, 1, 3]. In a conformal coordinate chart, s_z denotes the approximate metric differential of u at z on the coordinate tangent plane $\mathbb{R}^2$. Normalize KS energy by $I(s) = \pi^{-1} \int_{S^1} s(v)^2 dv$. For $A \in GL^+(2)$, set $F_s(A) = I(s \circ A)/\det A$. The angular substitution $w = Av/|Av|$, whose Jacobian is $\det A/|Av|^2$, yields

$$F_s(A) = \frac{1}{\pi(\det A)^2} \int_{S^1} \frac{s(w)^2}{|A^{-1}w|^4} dw. \quad (1)$$

This is smooth in A near the identity even for a nonsmooth or degenerate seminorm. Each fixed-order matrix derivative is bounded by $CM(s)^2$. Its first derivative is $DF_s(I)[D] = T(s) : D$, where

$$T(s) = \frac{1}{\pi} \int_{S^1} s(w)^2 (4w \otimes w - 2I) dw.$$

The stress is symmetric and tracefree. For a smooth vector field V supported inside a minimizing source chart, use $u \circ \phi_\varepsilon$ with $\phi_\varepsilon = \text{id} + \varepsilon V$. On a smaller domain enclosing its support this is an admissible same-trace competitor for either sign of ε . After the change of variables, the differential seminorm is held at the fixed new point and the matrix is $D\phi_\varepsilon \circ \phi_\varepsilon^{-1}$. The $CM(s)^2$ bound is integrable. Dominated differentiation gives

$$\int T(s_z) : DV(z) dz = 0.$$

Write $T = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Its distributional divergence equations are $a_x + b_y = 0$, $b_x - a_y = 0$, so $q = a - ib$ is holomorphic by the distributional Weyl lemma. Scaling and rotation give $T(\lambda s) = \lambda^2 T(s)$ and $T(s \circ R) = R^T T(s) R$. Under a holomorphic coordinate change $z = z(w)$, $q_w = (dz/dw)^2 q_z$. Thus $q_z dz^2$ is a global holomorphic quadratic differential. Each pole has its own minimizing chart and weak equation; no unproved puncture-removal claim is used. On the sphere this differential vanishes: its coefficient is entire in a plane chart and is $O(|z|^{-4})$ at infinity. Hence $T(s_z) = 0$ almost everywhere.

For a nonzero balanced seminorm normalize $M(s) = 1$. Choose e_2 with $s(e_2) = m$, and put $L = s(e_1)$ for its perpendicular unit vector. The triangle inequality gives $L^2 \geq 1 - m^2$. For $r(v) = L|v_1|$, the reverse triangle inequality gives $|s(v) - r(v)| \leq m$ on the unit circle. Both are at most one, so $|s(v)^2 - r(v)^2| \leq 2m$ and

$$\|T(s) - T(r)\|_{\text{op}} \leq 8m, \quad T(r) = L^2 \text{diag}(1, -1).$$

Consequently $1 - m^2 \leq L^2 \leq 8m$, and $m \geq \sqrt{17} - 4$. Undoing normalization proves $M/m \leq 4 + \sqrt{17}$; zero seminorms satisfy the inequality trivially. This proves Theorem 2.

For inner-product seminorms, $T(s) = 2G - \text{tr}(G)I$ and this recovers classical weak conformality [5, Lemma 1.5]. In general, fourfold-symmetric non-Euclidean norms can be balanced. Bounded distortion, not roundness, is the conclusion. The older, sharper KS bound also applies here. Indeed, $J(A) = I(s \circ A)$ is convex in the matrix A , and the angular identity gives $DJ(I)[D] = T(s) : D + I(s) \text{tr} D$. Once $T(s) = 0$, for any positive symmetric S with $\det S = 1$ convexity and $\text{tr} S \geq 2$ imply

$$J(S) \geq J(I) + I(s)(\text{tr} S - 2) \geq J(I).$$

Every $A \in SL(2)$ has polar decomposition $A = SR$, with R a rotation; circular invariance gives $J(A) = J(S)$. Thus the balanced seminorm is $SL(2)$ -minimal. The pointwise Lytchak–Wenger bound $2\sqrt{2} + \sqrt{6}$ applies [3, Lemma 3.1 and §3.3]. We retain the elementary $4 + \sqrt{17}$ proof for a self-contained bound; no new, improved, or optimal distortion constant is claimed.

Attribution, verification, and review status

The question and homotopy-minimizer theory belong to Meier–Vikman–Wenger. We credit Lytchak–Wenger’s general energy/area framework, Korevaar–Schoen’s Sobolev and NPC tensor theory, the classical Hopf/Sacks–Uhlenbeck method, McShane extension, and standard form/comass calibration. The compact belt/cap construction and the complete arguments above

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