A regular zero-area singularity with a nonzero weak Bianchi defect

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Abstract

We apply a negative-mass member of the known Dadhich–Kar–Mukherjee–Visser metric family to the regular-zero-area-singularity conjecture in Burtscher's Oberwolfach report. Its natural static slices have harmonically regular zero-area singularities of mass $-2a$, but the homogeneous weak second Bianchi identity fails under the boundary-reaching vector-test convention of Burtscher–Kiessling–Tahvildar-Zadeh. We prove absolute convergence of the full scalar contraction for every such compactly supported smooth test and compute its exact boundary functional; a normalized radial test gives -8π. The metric and its Einstein tensor are classical, and the flux method is already present in the cited weak-identity work. The contribution is this explicit counterexample application and its convergence check. The example violates the null energy condition. The conclusion concerns the geometric implication in the cited static-foliation framework, with no specified matter model; it asserts neither regularity on every arbitrary tilted slice nor an identity for tests smooth after collapsing the boundary sphere to a point.

1 The geometric question and the test convention

Burtscher's contribution to Oberwolfach Report 40/2021 describes admissible timelike singularities using regular zero-area singularities (ZAS) of negative mass on spacelike slices, and conjectures the homogeneous weak Einstein-tensor identity for that class [1, pp. 2232–2233]. The cited technical framework of Burtscher–Kiessling–Tahvildar-Zadeh (BKTZ) uses static slices and a retained inner boundary [2, Sections 1.3–1.4]. This note addresses that geometric implication with their boundary-reaching vector tests. The short report does not give a fully formal definition for every nonsymmetric spacetime or explicitly require regularity on every tilted hypersurface. We state the foliation interpretation explicitly rather than claim to settle every possible strengthening of its wording. The report's broader matter and evolution program remains a separate question.

Fix $a > 0$ and define

$$\overline{M} = \mathbb{R}_t \times \{x \in \mathbb{R}^3 : |x| \geq a\}, \qquad M = \operatorname{int} \overline{M}, \qquad \rho = |x|, \quad S_a = \{\rho = a\}, \quad \phi = 1 - \frac{a}{\rho}.$$

The smooth structure of $\overline{M}$ retains the sphere S_a. Our test space is $\mathcal{T} = C_c^\infty(\overline{M}; T\overline{M})$: smooth vector-field sections with compact support in the manifold-with-boundary topology. Their normal boundary component need not vanish. This is the convention in BKTZ, Definition 2.1, the boundary-support footnote to Theorem 1.1, and the radial term in equation (2.5) [2]. A section of the tangent bundle at a boundary is not required to be tangent to that boundary.

We use signature $(-+++)$, $c = G_{\text{Newton}} = 1$, and $G_{\mu\nu} = \operatorname{Ric}(g)_{\mu\nu} - \frac{1}{2} R_g g_{\mu\nu}$, with the Ricci convention

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa} - \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa}.$$

For a metric smooth on M, write

$$I_g(\psi) = \int_M G^\mu{}_\nu \nabla_\mu \psi^\nu \, dV_g, \qquad \psi \in \mathcal{T}, \tag{1}$$

whenever the full scalar contraction is absolutely integrable. On M this equals $\int_M G_{\mu\nu}\nabla^\mu\psi^\nu \, dV_g$ pointwise. Lowering a test vector with a singular metric need not preserve boundary smoothness; independently smooth covectors are not substituted for $\mathcal{T}$. The homogeneous weak identity is $I_g(\psi) = 0$ for every test in this specified class.

2 A classical metric and an explicit counterexample

The metric used below and the affine-lapse family used in its calculation are known. Dadhich–Kar–Mukherjee–Visser (DKMV) give [3, equations (9)–(15)]

$$ds^2 = -\left(\kappa + \lambda\frac{1 - m/(2\bar{r})}{1 + m/(2\bar{r})}\right)^2 dt^2 + \left(1 + \frac{m}{2\bar{r}}\right)^4 (d\bar{r}^2 + \bar{r}^2 d\Omega^2),$$

along with its Einstein-tensor components. The substitution

$$(m, \bar{r}, \kappa, \lambda) = (-2a, \rho, A, B), \qquad N_v = \frac{\rho + a}{\rho - a}, \qquad N = A + BN_v \tag{2}$$

gives $g_N = -N^2 dt^2 + \phi^4 \delta$, where δ is the Euclidean metric. We take $A, B \geq 0$, $A + B > 0$, so $N > 0$. DKMV's subsequent wormhole parameter classification takes $m > 0$; the negative-mass substitution here is an application of their metric formula, not an attribution of this later ZAS or weak-Bianchi conclusion to that classification. No new metric family or Einstein tensor is claimed. The spatial negative-mass Schwarzschild ZAS is also the standard example in Bray–Jauregui [4].

Theorem 1. *For $a > 0$, the metric*

$$g = -\phi^{-2} dt^2 + \phi^4 (d\rho^2 + \rho^2 d\Omega^2) \tag{3}$$

is static, spherically symmetric and asymptotically flat on M. Every slice of its natural static foliation is totally geodesic and has a harmonically regular ZAS at S_a of regular mass $-2a$. Its inner end is a genuine curvature singularity, timelike in the limiting radial conformal sense. For every $\psi \in \mathcal{T}$, the scalar in (1) *is absolutely integrable and*

$$I_g(\psi) = -\frac{2}{a^2} \int_{\mathbb{R}\times S_a} \langle \psi_{\text{spatial}}, n \rangle_\delta \, dt \, dA_\delta, \qquad n = \frac{x}{\rho}. \tag{4}$$

In particular there is a test in $\mathcal{T}$ with $I_g(\psi) = -8\pi$. The effective tensor $T = G/(8\pi)$ violates the null energy condition.

The theorem disproves the proposed implication from regular negative-mass ZAS slice geometry to the homogeneous identity under the stated foliation and test convention. A specified matter system or an added energy condition is not among its hypotheses. We prove the theorem through the following geometric and analytic calculations; no numerical verification is used as a proof of a quantified assertion.

2.1 Regular ZAS and the timelike singular end

The resolution (δ, ϕ) extends smoothly to S_a, where ϕ vanishes simply. The Euclidean unit normal into the exterior is $+\partial_\rho$, and

$$\partial_\rho \phi|_{S_a} = \frac{1}{a} > 0, \qquad \Delta_\delta \phi = 0.$$

These are the regular and harmonic resolution conditions of Bray–Jauregui, Definitions 6–7 [4]. Their ZAS area condition concerns approaching boundary graphs converging in C^1 (Definition 4). For every such family the Euclidean areas stay bounded, while the supremum of ϕ^4 on the surfaces tends to zero. Since $dA_h = \phi^4 dA_\delta$ for $h = \phi^4 \delta$, their physical areas tend to zero. This checks the full approaching-surface condition, not just coordinate spheres. Definition 10 then gives

$$m_{\mathrm{reg}} = -\frac{1}{4}\left(\frac{1}{\pi}\int_{S_a} (\partial_\rho \phi)^{4/3}\, dA_\delta\right)^{3/2} = -\frac{1}{4}(4a^{2/3})^{3/2} = -2a. \tag{5}$$

The time-independent spatial metric and zero shift give second fundamental form $K = 0$. These statements concern the natural static foliation.

The area radius satisfies

$$r = \rho\phi^2 = \frac{(\rho - a)^2}{\rho}, \qquad f = 1 + \frac{4a}{r} = N_v^2, \qquad h = f^{-1}dr^2 + r^2 d\Omega^2. \tag{6}$$

Here $dr/d\rho = 1 - a^2/\rho^2 > 0$ on M; the change of variable is used only on the smooth interior. At infinity $\phi^4 = 1 - 4a/\rho + O(\rho^{-2})$ and $\phi^{-1} = 1 + a/\rho + O(\rho^{-2})$, with the corresponding differentiated decay. Thus (3) is asymptotically flat.

For its radial causal geometry put

$$s(\rho) = \int_a^\rho \phi(u)^3\, du \sim \frac{(\rho - a)^4}{4a^3}.$$

The radial metric is $\phi^{-2}(-dt^2 + ds^2)$, so its limiting conformal boundary $s = 0$ is timelike. Each nearby constant-ρ tube has spacelike normal and Lorentzian induced metric. A radial null geodesic with conserved energy $E = \phi^{-2}dt/d\lambda > 0$ obeys $|d\rho/d\lambda| = E/\phi$; its remaining affine length is

$$\frac{1}{E}\int_a^\rho \phi(u)\, du = \frac{\rho - a - a\log(\rho/a)}{E} \sim \frac{(\rho - a)^2}{2aE}.$$

The invariant computed below diverges at this finite-affine-length end. The timelike statement is about this limiting causal structure; no nondegenerate smooth extension of the physical four-metric to the cylinder is asserted.

2.2 Curvature calculation

For completeness we recover the old tensor directly. For $h = f^{-1}dr^2 + r^2 d\Omega^2$, its spatial Ricci and radial-Hessian mixed eigenvalues are

$$(\mathrm{Ric}(h)^r{}_r, \mathrm{Ric}(h)^\theta{}_\theta, \mathrm{Ric}(h)^\varphi{}_\varphi) = \left(\frac{4a}{r^3}, -\frac{2a}{r^3}, -\frac{2a}{r^3}\right),$$

$$(\mathrm{Hess}_h\, u)^r{}_r = fu'' + \frac{f'}{2}u', \qquad (\mathrm{Hess}_h\, u)^\theta{}_\theta = (\mathrm{Hess}_h\, u)^\varphi{}_\varphi = \frac{fu'}{r}.$$

These follow from $\Gamma^r_{rr} = -f'/(2f)$, $\Gamma^r_{\theta\theta} = -fr$, $\Gamma^r_{\varphi\varphi} = -fr\sin^2\theta$, $\Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = 1/r$, and the usual unit-sphere connection. With $q = N_v = \sqrt{f}$,

$$q' = -\frac{2a}{qr^2}, \qquad q'' = \frac{4a}{qr^3} - \frac{4a^2}{q^3r^4}, \qquad \mathrm{Hess}_h\, q = q\,\mathrm{Ric}(h), \quad \Delta_h q = 0.$$

For $g_N = -N^2dt^2 + h$, the additional time connection gives

$$\mathrm{Ric}(g_N)_{tt} = N\Delta_h N, \quad \mathrm{Ric}(g_N)_{ti} = 0, \quad \mathrm{Ric}(g_N)_{ij} = \mathrm{Ric}(h)_{ij} - N^{-1}(\mathrm{Hess}_h\, N)_{ij}.$$

Using $N = A + Bq$ and $R_h = 0$, we obtain $R_{g_N} = 0$, all off-diagonal mixed Einstein entries zero, and

$$G^t{}_t = 0, \qquad (G^r{}_r, G^\theta{}_\theta, G^\varphi{}_\varphi) = \frac{A}{N}\left(\frac{4a}{r^3}, -\frac{2a}{r^3}, -\frac{2a}{r^3}\right). \tag{7}$$

These agree with DKMV's equations (9)–(11) under (2). The main lapse is $N = (1+q)/2 = \phi^{-1}$, so

$$(G^t{}_t, G^\rho{}_\rho, G^\theta{}_\theta, G^\varphi{}_\varphi) = \left(0, \frac{2a}{\rho^3\phi^5}, -\frac{a}{\rho^3\phi^5}, -\frac{a}{\rho^3\phi^5}\right). \tag{8}$$

Since $R_g = 0$, squaring the Ricci tensor yields

$$\mathrm{Ric}(g)_{\mu\nu}\,\mathrm{Ric}(g)^{\mu\nu} = \frac{6a^2}{\rho^6\phi^{10}} \longrightarrow +\infty \quad (\rho \downarrow a). \tag{9}$$

This is invariant curvature blow-up, despite the zero scalar curvature. In a static orthonormal frame the energy density is zero and each angular pressure is $-a/(8\pi\rho^3\phi^5) < 0$. An angular null vector with frame components $(1, 0, 1, 0)$ therefore has $T(k, k) < 0$, proving the stated NEC violation.

3 Absolute convergence and the exact defect

We establish the full boundary functional for the classical family; its main member gives (4). BKTZ already use a limiting boundary-flux calculation [2, Definition 2.1 and the calculation following equation (2.5)]. Here the necessary convergence is proved directly, without invoking their sufficient theorem outside its hypotheses.

Proposition 1. *For $a > 0$, $A, B \geq 0$, $A + B > 0$, let g_N be defined by (2). For every $\psi \in \mathcal{T}$ the full contraction is absolutely integrable and*

$$I_{g_N}(\psi) = -\frac{4A}{a^2}\int_{\mathbb{R}\times S_a} \langle \psi_{\mathrm{spatial}}, n\rangle_\delta \, dt\, dA_\delta. \tag{10}$$

Thus the homogeneous identity for every test in $\mathcal{T}$ holds within this family exactly when $A = 0$.

Proof. In exterior Cartesian coordinates $dV_{g_N} = N\phi^6\, dt\, d^3x$. Equation (7) and $r^3 = \rho^3\phi^6$ give the weighted mixed spatial tensor

$$D^i{}_j := N\phi^6 G^i{}_j = \frac{Aa}{\rho^3}(6n_in_j - 2\delta_{ij}). \tag{11}$$

It is Euclidean symmetric, traceless, and smooth up to $\rho = a > 0$. Writing $\phi_i = \partial_i\phi$, the spatial connection is

$$\Gamma^j_{ik} = \frac{2}{\phi}(\delta^j_i\phi_k + \delta^j_k\phi_i - \delta_{ik}\phi^j), \qquad D^i{}_j\Gamma^j_{ik} = \frac{2}{\phi}((\mathrm{tr}\, D)\phi_k + D^i{}_k\phi_i - D^k{}_j\phi_j) = 0.$$

The time row and column of $G^{\mu}{}_{\nu}$ vanish, and $\Gamma^{j}_{it} = 0$. Hence, pointwise on M,

$$G^{\mu}{}_{\nu}\nabla_{\mu}\psi^{\nu}\, dV_{g_N} = D^{i}{}_{j}\partial_i\psi^{j}\, dt\, d^3x. \tag{12}$$

On the compact support of any $\psi \in \mathcal{T}$, D and the Cartesian first derivatives of ψ are bounded, including at the boundary. The support has finite Euclidean measure. Thus the absolute value of the right side is integrable. This proves convergence for arbitrary time and angular dependence and avoids spherical-coordinate pole issues.

The radial and tangential eigenvalues of D are $D_r = 4Aa/\rho^3$ and $D_T = -2Aa/\rho^3$. Consequently

$$\partial_i D^{i}{}_{j} = \left(D_r' + \frac{2}{\rho}(D_r - D_T)\right) n_j = 0.$$

Integrate (12) by parts over $a+\varepsilon < \rho < R$, with R and the time endpoints outside the support. The inner outward Euclidean normal of this exterior domain is $-n$. The only boundary contribution is

$$-\frac{4Aa}{(a+\varepsilon)^3}\int_{\mathbb{R}\times S_{a+\varepsilon}} \langle \psi_{\text{spatial}}, n\rangle_{\delta}\, dt\, dA_{\delta}.$$

Absolute convergence of the interior integral and smoothness of the boundary trace permit $\varepsilon \downarrow 0$, proving (10). At $A = 0$ the Einstein tensor is identically zero. For $A > 0$ the radial test below has nonzero functional, proving the final assertion. □

Choose $\eta \in C_c^{\infty}(\mathbb{R})$ with $\int \eta\, dt = 1$ and $\chi \in C_c^{\infty}([a,\infty))$ equal to 1 near a. Set

$$\psi = \eta(t)\chi(\rho)\partial_{\rho}. \tag{13}$$

Since $\partial_{\rho} = x^i\partial_{x^i}/\rho$ and $a > 0$, this is smooth on $\overline{M}$ and has a nonzero normal boundary trace. Its spherical components are angular independent, meeting also the convenient restriction used in BKTZ's equation (2.5). Formula (10) gives $I_{g_N}(\psi) = -16\pi A$, hence -8π at $A = B = 1/2$. As a direct sign and density check, its complete scalar density is

$$4Aa\eta(t)\left(\frac{\chi'(\rho)}{\rho} - \frac{\chi(\rho)}{\rho^2}\right) dt\, d\rho\, d\Omega = 4Aa\eta(t)\partial_{\rho}(\chi/\rho)\, dt\, d\rho\, d\Omega.$$

The boundary endpoint is $\chi(a)/a = 1/a$, which gives the same negative sign. This completes the proof of Theorem 1.

Individual connection summands in this radial computation can each have logarithmically divergent integrals. Equation (12) combines the ordinary smooth summands on M before integration. It does not separately integrate divergent terms, take a principal value, multiply tensor distributions, or assign a nonsingular boundary value to the physical metric.

4 Scope, checks, and disclosure

The counterexample has nonzero order $(\rho - a)^{-5}$ mixed Einstein components. In particular, with $x = \rho - a$, $x^5 G^{\rho}{}_{\rho} \to 2a^3 \neq 0$. BKTZ Theorem 1.1 assumes the strictly improved bound $O(x^{-5+\kappa})$ for some $\kappa > 0$ [2]; it does not cover this borderline case. The old boundary-flux method and the independent convergence proof are distinct from an invocation of that sufficient theorem. The vacuum control $A = 0, B > 0$ is the negative-mass Schwarzschild vacuum spacetime after time rescaling and has zero functional. Sharing its spatial metric does not make the main lapse vacuum.

The static-slice condition is essential to the claim's formulation. Bray–Jauregui use a time-symmetric setting and discuss extensions beyond it separately [4, p. 3 and Section 6.6 of the arXiv version]. For example, a uniformly spacelike local tilt $t = t_0 + qs(\rho)$, $|q| < 1$, has induced metric

$$h_q = \phi^4((1-q^2)d\rho^2 + \rho^2 d\Omega^2), \qquad m_{\mathrm{reg}}(h_q) = -\frac{2a}{1-q^2} < 0.$$

Its resolution remains smooth and positive. Arbitrary nearly null tilts need not have that property: $t = u(\rho)$ with $u' = \phi^3\sqrt{1-\phi^2}$ is interior-spacelike but has induced metric $\phi^6 d\rho^2 + \phi^4\rho^2 d\Omega^2$. Any simple defining conformal factor in the retained cylinder structure leaves a zero radial eigenvalue in the resolved boundary metric. No all-arbitrary-tilted-slices assertion is made.

Tests confined to the open interior give the ordinary classical identity, and tests smooth after collapsing S_a to a point have a different smoothness requirement. In particular the radial test (13) must not be replaced through that collapse. The effective matter violates NEC, so neither an energy-constrained version nor a specified electromagnetic matter system is resolved. The conclusion is the failure of the geometric implication in the stated boundary/static framework.

The accompanying archive preserves the thirteen available historical sources and supplies explicitly guarded normal/optimized reproductions and independent exact consistency checks. Its provenance note distinguishes eleven verified payload references from one unavailable, unattested standalone manifest digest. The analytic proof establishes the quantified conclusion; finite computations provide supporting falsification evidence, not proof by sampling.

A bounded independent literature search found no earlier printed application of this metric resolving the exact report conjecture in the inspected routes. This is not a certificate of historical first priority or continued openness. The metric, tensor, spatial ZAS example and boundary-flux method are attributed to the sources above. The supporting priority ledger records search coverage and its limits.

AI use and review status. AI tools were used extensively in developing, writing, reproducing and adversarially checking this work, including parallel AI reviews. This is an unrefereed research note. No independent human peer review or formal proof-assistant verification is claimed.

References

[1] A. Burtscher, *Matter particles, naked singularities, and a weak second Bianchi identity*, in *Mathematical Aspects of General Relativity*, Oberwolfach Reports **18** (2021), no. 3, Report 40, pp. 2231–2233. doi:10.4171/OWR/2021/40. The volume/workshop year is 2021; the publisher lists publication on 26 November 2022.

[2] A. Burtscher, M. K.-H. Kiessling and A. S. Tahvildar-Zadeh, *Weak second Bianchi identity for static, spherically symmetric spacetimes with timelike singularities*, Classical and Quantum Gravity **38** (2021), 185001. doi:10.1088/1361-6382/ac1853; arXiv:1901.00813v3. Numbered locations in this note refer to that arXiv version.

[3] N. Dadhich, S. Kar, S. Mukherjee and M. Visser, $R = 0$ *spacetimes and self-dual Lorentzian wormholes*, Physical Review D **65** (2002), 064004. doi:10.1103/PhysRevD.65.064004; arXiv:gr-qc/0109069 (2001). Published 7 February 2002. Equation numbers refer to the original arXiv manuscript.

[4] H. L. Bray and J. L. Jauregui, *A geometric theory of zero area singularities in general relativity*, Asian Journal of Mathematics **17** (2013), 525–559; arXiv:0909.0522v2. Definition and section locations refer to that arXiv version.