

Infinite modulus between a compact real set and its real complement

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Abstract

Let $E \subset [0, 1]$ be compact and let $F = \mathbb{R} \setminus E$ be its complement within the real axis. We prove that positive planar 2-modulus of the family joining E to $\{|z| = 2\}$ implies infinite modulus of the family joining E to F . Admissibility is imposed on every curve for pointwise extended Borel densities. A weighted-distance potential and a critical trace estimate first force any hypothetical finite-energy separator to have $\mathcal{L}^1(E) = 0$; a separate small-circle argument then treats every such null set, including null sets of positive conformal capacity. The result is stated under explicit conventions for the question recorded as Hayman–Lingham Problem 7.58.

1 Statement and conventions

We identify $\mathbb{R}$ with the horizontal axis in $\mathbb{C} = \mathbb{R}^2$, write dA for planar Lebesgue measure, and put $S_2 = \{z : |z| = 2\}$. A connector from A to B is a continuous curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ with $\gamma(0) \in A$, $\gamma(1) \in B$, locally rectifiable on $(0, 1)$. Equivalently, one may use the open parameter interval and require the stated endpoint limits. For a nonnegative Borel function ρ , its line integral is the increasing limit of the integrals on compact interior subcurves. Arclength is counted with multiplicity; constant portions have zero length. In particular, an infinite value at an endpoint alone contributes no mass. The integral may be infinite, and the whole curve need not have finite Euclidean length.

Let $\Gamma(A, B)$ consist of *all* such connectors in the whole plane. There is no condition that their interiors avoid either plate. Curves may cross or revisit A , B , or the real axis before their terminal endpoints. Define

$$\begin{aligned} \text{Mod}_2 \Gamma &= \inf_{\rho \in \text{Adm}(\Gamma)} \int_{\mathbb{R}^2} \rho^2 dA, \\ \text{Adm}(\Gamma) &= \left\{ \rho : \mathbb{R}^2 \rightarrow [0, \infty] \text{ Borel} : \right. \\ &\quad \left. \int_{\gamma} \rho ds \geq 1 \quad \text{for every } \gamma \in \Gamma \right\}. \end{aligned}$$

The infimum of an empty set is ∞ . Densities are pointwise functions, not equivalence classes modulo area-zero sets, and may be infinite on such sets. We also prove the theorem when $\Gamma(A, B)$ is defined using only globally rectifiable connectors, provided that convention is used consistently in both modulus families.

Theorem 1. *For every compact $E \subset [0, 1]$, with $F = \mathbb{R} \setminus E$, one has*

$$\text{Mod}_2 \Gamma(E, S_2) > 0 \implies \text{Mod}_2 \Gamma(E, F) = \infty.$$

The complement here is the real complement, not the planar complement. Hayman and Lingham record the question, attributed to A. A. Gonchar and communicated by M. Vuorinen, in Problem 7.58 of the 2018 version of their collection [3]. The theorem specifies the curve and density conventions used here; no conclusion about other interpretations of unspecified source terminology is intended.

We prove the contrapositive. The intermediate conclusion $\mathcal{L}^1(E) = 0$ is not sufficient: a compact null subset of the real axis can have positive conformal capacity. Section 4 supplies the additional argument at every endpoint of E .

2 A pointwise separating potential

The following majorization is pointwise, including at infinite values.

Lemma 1. *If Q is a compact rectangle and $r : Q \rightarrow [0, \infty]$ is Borel with $\int_Q r^2 dA < \infty$, there is a relatively lower semicontinuous $g : Q \rightarrow [1, \infty]$ such that $g \geq r$ everywhere and $\int_Q g^2 dA < \infty$.*

Proof. Set $h = r^2$ and $A_k = \{h > 2^k\}$, $k \geq 0$. By outer regularity choose relatively open $O_k \supset A_k$ with $|O_k| \leq |A_k| + 2^{-2k-4}$, where $|\cdot|$ denotes planar area in this proof. Put

$$H = 1 + \sum_{k \geq 0} 2^{k+1} \mathbf{1}_{O_k}, \quad g = \sqrt{H}.$$

Open-set indicators, finite sums, and increasing limits are lower semicontinuous. Thus g is lower semicontinuous. Dyadic geometric sums give $H \geq h$ at every point: the constant term covers $h \leq 1$, and if $h = \infty$, the point belongs to every O_k . Moreover, $\sum_{k \geq 0} 2^{k+1} \mathbf{1}_{\{h > 2^k\}} \leq 4h$ for finite $h \geq 0$. Tonelli therefore gives

$$\int_Q g^2 dA \leq |Q| + 4 \int_Q r^2 dA + \sum_{k \geq 0} 2^{-k-3} = |Q| + 4 \int_Q r^2 dA + \frac{1}{4} < \infty.$$

In particular, any infinite values retained by g form an area-zero set. □

Fix $Q = [-2, 3] \times [-2, 2]$. For a nonempty compact $E \subset [0, 1]$ and a lower semicontinuous $g \geq 1$ on Q with finite squared integral, define

$$d(z) = \inf_{\alpha: E \rightarrow z, \alpha \subset Q} \int_{\alpha} g ds, \quad u(z) = \min\{1, d(z)\}, \quad (1)$$

where the auxiliary paths are globally rectifiable. The infimum is ∞ if there is no finite-cost path. Constant paths at points of E are allowed.

Lemma 2. *The functions d and u are lower semicontinuous. For every rectifiable path $\beta \subset Q$ from x to y ,*

$$|u(x) - u(y)| \leq \int_{\beta} g ds. \quad (2)$$

The actual pointwise function u is absolutely continuous on almost every horizontal and vertical line in Q , and $u \in W^{1,2}(\text{int } Q)$.

Proof. We first justify the weighted-length compactness needed for measurability. For a nonnegative continuous weight a on Q and a rectifiable curve $\alpha : [0, 1] \rightarrow Q$, consider the partition sums

$$S_P(\alpha) = \sum_i \min_{t \in [t_{i-1}, t_i]} a(\alpha(t)) |\alpha(t_i) - \alpha(t_{i-1})|.$$

Each S_P is continuous under uniform convergence of curves. Its supremum over finite partitions is $\int_\alpha a ds$. Indeed, fine partitions make the oscillation of $a \circ \alpha$ arbitrarily small, while further refinement approximates the total variation of α by chord sums. The resulting error is bounded by the weight oscillation times the curve length plus the chord-variation error times $\max_Q a$. Thus weighted length for a is lower semicontinuous under uniform convergence.

A nonnegative lower semicontinuous g on compact Q is the increasing pointwise limit of continuous nonnegative functions; explicitly one may use

$$a_m(x) = \min \left\{ m, \inf_{y \in Q} (g(y) + m|x - y|) \right\}.$$

These functions are continuous and increasing. If their limit at x were strictly below $g(x)$, bounded-cost near-minimizers would tend to x , contrary to lower semicontinuity. This also proves convergence where $g(x) = \infty$. Monotone convergence of line integrals now shows that weighted length for g is lower semicontinuous as well.

Suppose $z_j \rightarrow z$ and $L = \liminf_j d(z_j) < \infty$. Pass to a subsequence with $d(z_j) \rightarrow L$, and choose paths α_j in (1) of cost at most $d(z_j) + 1/j$. Since $g \geq 1$, their Euclidean lengths are bounded. Constant-speed parametrization on $[0, 1]$ gives a common Lipschitz bound. Arzelà–Ascoli supplies a uniformly convergent subsequence in compact Q . Its limit α is rectifiable, starts in compact E , ends at z , and satisfies

$$d(z) \leq \int_\alpha g ds \leq \liminf_j \int_{\alpha_j} g ds \leq L.$$

Hence d and its truncation u are lower semicontinuous and Borel. This argument also covers infinite distances without assuming that the infimum is attained.

Concatenation with β gives the distance triangle inequality whenever the costs on its right side are finite. Reversing x, y and truncating yields (2). If one distance is infinite and the other finite, any finite-cost connection would make both finite; if both are infinite, both truncated values are 1. Thus the same inequality holds in all cases.

By Fubini and Cauchy–Schwarz, g is integrable on almost every coordinate line in Q . On each such line (2) holds for every subsegment. Applied to disjoint subintervals, it proves absolute continuity of the actual function u , with the coordinate derivative bounded by g almost everywhere. Slice integration by parts against a test function, followed by Fubini, identifies these derivatives as weak derivatives. Since $0 \leq u \leq 1$ and $|\nabla u|^2 \leq 2g^2$ almost everywhere, $u \in W^{1,2}(\text{int } Q)$. $\square$

Now suppose that a finite-energy Borel density ρ is admissible for $\Gamma(E, F)$. Apply Lemma 1 to its restriction to Q . Every auxiliary path ending at $F \cap Q$ is a member of the original connecting family. Its g -cost is therefore at least 1. Constant paths at E have cost zero, so the potential has the exact point values

$$u = 0 \text{ on } E, \quad u = 1 \text{ on } F \cap Q. \quad (3)$$

These values follow from admissibility for every curve. No arbitrary modification of a Sobolev representative on the real axis is being made.

Choose $\eta \in C_c^\infty(\text{int } Q)$ equal to 1 near $[0, 1] \times \{0\}$, and extend $w = \eta(1 - u)$ by zero to $\mathbb{R}^2$. Then $w \in W^{1,2}(\mathbb{R}^2)$ has compact support. Its actual restrictions to almost every vertical line are absolutely continuous. By (3) their values at height zero give

$$f(x) = w(x, 0) = \mathbf{1}_E(x) \quad \text{for almost every } x \in \mathbb{R}. \quad (4)$$

The exceptional vertical lines have one-dimensional measure zero. No capacity statement about their intersection with E is inferred.

3 The critical trace forces length zero

We give the required special cases of the standard flat trace and critical characteristic-function rigidity principles; compare [2, Propositions 3.4 and 3.8] and [1, Corollaries 1–2 and Remark 3].

Lemma 3. *For compactly supported $w \in W^{1,2}(\mathbb{R}^2)$, let f be its almost-everywhere vertical absolute-continuity trace. Then*

$$\int_{\mathbb{R}} \frac{\|f(\cdot + h) - f\|_{L^2(\mathbb{R})}^2}{h^2} dh < \infty. \quad (5)$$

Proof. First take w smooth and compactly supported, with unitary horizontal Fourier transform $W(\xi, y) = \widehat{w(\cdot, y)}(\xi)$, using $e^{-ix\xi}$ as frequency convention. Integration of the derivative of $|W|^2$ on $y > 0$, followed by $2ab \leq a^2 + b^2$, gives

$$|\xi| |W(\xi, 0)|^2 \leq 2|\xi| \int_0^\infty |W| |\partial_y W| dy \leq \int_0^\infty (\xi^2 |W|^2 + |\partial_y W|^2) dy.$$

Consequently Plancherel implies

$$\int_{\mathbb{R}} |\xi| |\widehat{f}(\xi)|^2 d\xi \leq \int_{y>0} |\nabla w|^2 dA. \quad (6)$$

There is no division by ξ , including at zero frequency.

For a general compactly supported w , mollification gives compactly supported smooth $w_j \rightarrow w$ in $W^{1,2}$. The traces converge to the actual vertical trace in L^2 . To see this directly, absolute continuity on almost every vertical line gives, for each such compactly supported v ,

$$|v(x, 0)|^2 \leq \int_0^\infty (|v(x, y)|^2 + |\partial_y v(x, y)|^2) dy.$$

This follows by integrating $\partial_y |v|^2$ and using $2|v||v'| \leq |v|^2 + |v'|^2$. Apply it to $v = w_j - w$ and integrate in x . Plancherel gives L^2 convergence of the trace Fourier transforms; a subsequence converges almost everywhere. Fatou then extends (6) to w .

Finally Plancherel, Tonelli, and the substitution $t = \xi h$ give

$$\int_{\mathbb{R}} \frac{\|f(\cdot + h) - f\|_2^2}{h^2} dh = C \int_{\mathbb{R}} |\xi| |\widehat{f}(\xi)|^2 d\xi, \quad C = \int_{\mathbb{R}} \frac{|e^{it} - 1|^2}{t^2} dt \in (0, \infty).$$

The defining integrand for C is bounded near zero and at most $4/t^2$ away from zero. This proves (5). $\square$

Lemma 4. *If a measurable $E \subset [0, 1]$ has $\mathbf{1}_E$ satisfying (5), then $\mathcal{L}^1(E) = 0$.*

Proof. Write $f = \mathbf{1}_E$. For $0 < h < 1$, put $N = \lceil 2/h \rceil \leq 3/h$. Since $Nh \geq 2$, the supports of f and $f(\cdot + Nh)$ are disjoint. Telescoping N translations in L^1 yields

$$2\mathcal{L}^1(E) = \|f(\cdot + Nh) - f\|_1 \leq N\|f(\cdot + h) - f\|_1.$$

Characteristic-function differences satisfy $|f(x+h) - f(x)|^2 = |f(x+h) - f(x)|$. Hence

$$\|f(\cdot + h) - f\|_2^2 \geq \frac{2}{3}\mathcal{L}^1(E)h.$$

The integral in (5), restricted to $0 < h < 1$, is at least $\frac{2}{3}\mathcal{L}^1(E) \int_0^1 dh/h$. Its finiteness forces $\mathcal{L}^1(E) = 0$. $\square$

Applying these lemmas to (4) proves

$$\rho \in \text{Adm } \Gamma(E, F), \quad \int_{\mathbb{R}^2} \rho^2 dA < \infty \quad \implies \quad \mathcal{L}^1(E) = 0. \quad (7)$$

This argument includes positive-length sets without any interval. It gives no capacity conclusion for the remaining null sets.

4 Small circles at every endpoint

Lemma 5. *Let $E \subset [0, 1]$ have length zero. If a Borel $\rho \in L^2(\mathbb{R}^2)$ is admissible for $\Gamma(E, \mathbb{R} \setminus E)$, set $q = \rho + \mathbf{1}_{B(0,3)}$. Then*

$$\int_{\gamma} q ds = \infty \quad \text{for every } \gamma \in \Gamma(E, S_2). \quad (8)$$

Proof. The function q is Borel, belongs to $L^2(\mathbb{R}^2)$, and remains admissible for $\Gamma(E, F)$. Suppose a connector from a particular $e \in E$ to S_2 has finite q -cost. Stop it at its first hit of S_2 . Its preceding image lies in $B(0, 2)$, where $q \geq 1$. Thus the lengths of all compact interior subcurves are bounded by its finite q -cost. Continuity and the endpoint limits show that the stopped curve has finite total variation, including at the initial endpoint. It is therefore a globally rectifiable curve and admits an arclength parametrization $\gamma : [0, L] \rightarrow \overline{B(0, 2)}$ after constant portions are removed. Its line integral agrees with the increasing interior-subcurve integral: endpoints have zero arclength measure. In particular $q \circ \gamma$ is integrable on $[0, L]$. This explains why no infinite-length endpoint behavior is lost in passing to a finite-cost curve. Its terminal point is at distance at least 1 from e .

Keep this actual endpoint e fixed. Define the extended circle energy

$$A_e(r) = \int_{|z-e|=r} q(z)^2 ds.$$

Polar integration gives $\int_0^\delta A_e(r) dr < \infty$ for every finite δ . Also $e + r \notin E$ for almost every $r > 0$, because $\mathcal{L}^1(E) = 0$. There are therefore radii $r_j \downarrow 0$, $r_j < 1$, with

$$e + r_j \in F, \quad r_j A_e(r_j) \longrightarrow 0. \quad (9)$$

For completeness, the radii with $e + r \in F$ and $A_e(r) < \infty$ have full measure. If, for some $\varepsilon, \delta > 0$, all such $0 < r < \delta$ satisfied $r A_e(r) \geq \varepsilon$, then $\int_0^\delta A_e(r) dr$ would dominate $\varepsilon \int_0^\delta dr/r = \infty$. Thus arbitrarily small good radii have arbitrarily small $r A_e(r)$, and a decreasing sequence can be chosen

recursively. This selection is made separately for each fixed endpoint; no exceptional set of points of E is discarded.

Let t_j be the first arclength parameter with $|\gamma(t_j) - e| = r_j$, and put $p_j = \gamma(t_j)$. Such parameters exist since the terminal distance is at least 1. Moreover $t_j \rightarrow 0$. Indeed, an arclength-parametrized curve is nonconstant on every initial interval $[0, t]$ of positive length. Some point of that interval has positive distance from e , so every sufficiently small disk centered at e is exited before t . Absolute continuity of the finite cost integral therefore gives

$$\int_{\gamma|_{[0, t_j]}} q \, ds = \int_0^{t_j} q(\gamma(s)) \, ds \longrightarrow 0. \quad (10)$$

Returns to e , retracing, or winding do not affect this first-exit argument.

Join p_j to the real point $e + r_j \in F$ along a circular arc centered at e . Cauchy–Schwarz on the full circle bounds the added cost by

$$\int_{|z-e|=r_j} q \, ds \leq (2\pi r_j A_e(r_j))^{1/2} \longrightarrow 0. \quad (11)$$

The initial portion and this arc concatenate to an actual globally rectifiable connector from E to the original real complement F . Interior plate contacts are allowed by definition. By (10)–(11), its q -cost is eventually less than 1, contradicting admissibility. This proves (8) for every connector, under either stated curve convention. $\square$

Proof of Theorem 1. For empty E , the family to S_2 is empty and has modulus zero. Otherwise suppose $\text{Mod}_2 \Gamma(E, F) < \infty$. By the definition of the infimum there exists an admissible Borel ρ of finite squared integral; no extremal density is needed. Equation (7) gives $\mathcal{L}^1(E) = 0$. Lemma 5 supplies $q \in L^2(\mathbb{R}^2)$ whose cost on every E -to- S_2 connector is infinite. Consequently q/n is admissible for the entire family for every positive integer n , and

$$0 \leq \text{Mod}_2 \Gamma(E, S_2) \leq n^{-2} \int_{\mathbb{R}^2} q^2 \, dA \longrightarrow 0.$$

Infinite-cost curves are included in this assertion, rather than removed as an exceptional family. The contrapositive proves the theorem. $\square$

5 Context and status

The critical trace and rigidity principles are classical [2, 1]. Pointwise majorants, weighted-distance potentials, and all-curve upper-gradient inequalities also occur in the established framework of Heinonen–Koskela [4]; no novelty of these mechanisms is asserted. The present proof supplies the needed pointwise construction and the separate null-set argument without importing a touching-plate capacity identity or quasicontinuous boundary values.

The accompanying research audit also verifies a complete indirect deduction assembled in October 2026 from older compact-condenser and Sobolev theory, starting with compact-plate results such as Hesse’s [5]. That assembly is not evidence that an earlier author printed the exact resolution. No earlier explicit resolution was authenticated in the bounded search, and historical priority remains undetermined. In particular, the historical report in the 2018 source does not establish present openness. The complete problem page in the 2019 Springer edition was unavailable for inspection, and the relevance of a 1982 Goldstein–Jackson conference item entitled “A problem of A. A. Gončar” remains unresolved. No firstness, current-openness, or new-mechanism claim is made here.

Disclosure. AI systems were used extensively in solving, proof drafting, source checking, and adversarial verification. This is an unrefereed research note. Its internal mathematical audits are not external human peer review, journal acceptance, or a formal proof-assistant certificate. The proof uses basic measure theory, Arzelà–Ascoli, elementary Sobolev mollification, and Plancherel; no computation is offered as a substitute for the analytic argument.

References

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