

Polynomial-time optimization and exact separation for rational mixed-integer bipartite covers

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Abstract

We give polynomial-time exact linear optimization, strong separation and membership for mixed-integer bipartite covers with arbitrary rational edge demands, unrestricted variable signs and any selected integer coordinates. All data and query points are rational in the total binary-input model; dimension and demand denominators vary. This resolves the finite rational interpretation of Conforti’s 2008 hull-membership problem. A direct specialized proximity proof localizes integer labels to intervals of length $2n$ and proves attainment. Exact continuous completion yields a submodular value oracle on a polynomial-size threshold lattice. A bounded-core argument establishes closedness of the actual hull and a polynomial facet-encoding bound, allowing classical optimization–separation equivalence to apply. The result is an explicit application of classical mixed proximity, discrete-convex and submodular-minimization ideas. It does not prove the separate subtree-hull conjecture or assert a compact formulation or strongly polynomial total algorithm.

1 Problem and contribution

Let U, V be specified disjoint vertex classes of a finite bipartite graph, $n = |U| + |V|$, $E \subset U \times V$, $I \subset U \cup V$, and $b \in \mathbb{Q}^E$. Define

$$S = \{x \in \mathbb{R}^n : x_u + x_v \geq b_{uv} \ (uv \in E), \ x_i \in \mathbb{Z} \ (i \in I)\}.$$

There is no nonnegativity assumption. Numerators, positive denominators, graph and integer-coordinate indices are encoded explicitly in binary. Conforti [1, pp. 2904–2905, Problem 1] asks the complexity of membership in $\text{conv}(S)$ and discusses linear optimization. Since that report leaves encoding of real query points implicit, our theorem specifies rational query points in the standard finite bit model.

Theorem 1. *For the data above and rational objective c , in polynomial total binary-input time one can return an exact rational minimizer of $c^T x$ on S , or a feasible point and an integral improving recession direction if the infimum is $-\infty$. The actual hull $\text{conv}(S)$ is a closed rational polyhedron with a polynomially computable facet-encoding bound. Strong separation and exact membership at every rational query, including boundary queries, are polynomial-time tasks. All n , $|I|$ and denominators may vary.*

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The zero-dimensional case is immediate; assume $n \geq 1$ below. Empty edge or integer-coordinate sets are allowed. Feasibility follows by setting every coordinate to a common integer at least zero and all demands. If I is empty, ordinary rational LP gives optimization directly; the hull argument below still applies.

Mixed proximity is classical: Cook–Gerards–Schrijver–Tardos [2, p. 258, Remarks (1)] give all-coordinate distance at most $n\Delta$ for integral combined constraint matrices, assuming a mixed optimum exists. Our signed-threshold proof is a direct specialization, and supplies attainment without that premise. Hybrid partial minimization and discrete-convex optimization are prior mechanisms [3, Proposition 2]; [4, §4.1] documents an earlier n^2 -element ring-family lineage. We claim no new general proximity or abstract ring-minimization method. The contribution is the explicit rational application, exact LP completion oracle, denominator-independent label count and actual-hull bit accounting.

Earlier network and bipartite-cover formulations [5, 6] depend polynomially on a numerical common denominator and explicitly discuss the remaining binary-size gap. They do not justify enumerating denominator residues here. A bounded source audit located no earlier full solution to this exact claim in inspected bodies; exhaustive priority is unverified.

2 Difference variables and proximity

Put $y_u = x_u$ for $u \in U$, $y_v = -x_v$ for $v \in V$, and transform c to a by the same sign change. This integral invertible map preserves the selected integer coordinates. Write

$$P = \{y : y_u - y_v \geq b_{uv}\}, \quad T = P \cap \{y : y_i \in \mathbb{Z} \ (i \in I)\}.$$

Rational LP decides boundedness. If $a^T y$ is unbounded on P , rational recession theory supplies a polynomial-bit rational solution of

$$r_u - r_v \geq 0 \quad (uv \in E), \quad a^T r \leq -1.$$

Clearing coordinate denominators produces a polynomial-bit integral improving ray. From the integral feasible point, every nonnegative integer step stays in T . Thus boundedness on P and T agrees. Otherwise rational LP returns a polynomial-bit minimizer y^* on P . No vertex of this unanchored, generally nonpointed polyhedron is assumed.

Lemma 2. *If a is bounded below on P , the minimum on T is attained, and at least one minimizer z satisfies $\|z - y^*\|_\infty < n$.*

Proof. Every $d \in \mathbb{R}^n$ has a signed nested-threshold decomposition $d = \sum_{h=1}^s \lambda_h r^h$, with $s \leq n$, $\lambda_h > 0$ and nonzero $r^h \in \{0, 1\}^n$ or $\{0, -1\}^n$. For positive coordinates, list distinct values $0 < t_1 < \dots < t_j$, use indicators of $\{i : d_i \geq t_h\}$ and successive differences $t_h - t_{h-1}$; do the same with negative magnitudes and negative indicators. Nesting makes every layer conformal on coordinates and differences: r_i^h has the sign of d_i or is zero, and $r_i^h - r_j^h$ has the sign of $d_i - d_j$ or is zero. Consequently each weighted coordinate and difference lies between zero and the corresponding value of d .

Take $z \in T$ and $d = z - y^*$. For each layer (λ, r) , the point $y^* + \lambda r$ lies in P , rowwise between y^* and z . Optimality gives $a^T r \geq 0$. If $\lambda \geq 1$, the point $z - r$ also lies rowwise between those endpoints, stays mixed-integer, has no greater objective and has strictly smaller ℓ_1 distance to y^* .

To avoid assuming attainment, fix any $z^0 \in T$. The closed nonempty sublevel set $A = \{z \in T : a^T z \leq a^T z^0\}$ has a distance minimizer: intersect it with the closed ℓ_1 ball of radius $\|z^0 - y^*\|_1$.

Compactness and the removal step rule out every layer of weight at least one at this minimizer. With at most n layers, its sup-distance is less than n . Hence every point of T has a no-more-expensive representative in the same compact box $y^* + [-n, n]^n$. Minimization on its closed mixed subset attains the global optimum, and a distance-minimizing optimum has the asserted strict bound. $\square$

3 Exact partial minimization on a short-label lattice

For $i \in I$ set

$$l_i = \lceil y_i^* - n \rceil, \quad h_i = \lfloor y_i^* + n \rfloor, \quad D = \{p \in \mathbb{Z}^I : l_i \leq p_i \leq h_i, \ p_u - p_v \geq \lceil b_{uv} \rceil \text{ if } u, v \in I\}.$$

Each width is at most $2n$. Exactly these assignments have completions in P : necessity is immediate; for sufficiency undo the sign change on fixed coordinates and set every remaining original coordinate to a sufficiently large common integer. Only edges between two integer coordinates are rounded. The free-sign bipartite model makes this completion possible. By Lemma 2, D is nonempty.

For $p \in D$, the slice value

$$f(p) = \min\{a^T y : y \in P, \ y_i = p_i \ (i \in I)\}$$

is finite, attained and computable exactly by polynomial-size rational LP. Both P and D are closed under coordinatewise meet and join. For example, two inequalities $y_u \geq y_v + b$, $z_u \geq z_v + b$ imply their min and max counterparts. Optimal completions y, z of p, q give

$$f(p) + f(q) = a^T y + a^T z = a^T (y \wedge z) + a^T (y \vee z) \geq f(p \wedge q) + f(p \vee q).$$

Thus f is submodular on the finite distributive lattice D . This closure is in difference variables; it would be false for original cover variables.

Uniform rational encoding. Let $B = \max(\{1\} \cup \{\lceil |b_e| \rceil : e \in E\})$, $A_0 = \max(1, \max_i \lceil |a_i| \rceil)$, $H^* = \lceil \|y^*\|_\infty \rceil + n$, and let k, q be the lcms of demand and objective denominators (empty lcm is one). Every p has magnitude at most H^* . Anchor one coordinate at zero in each slice component with no fixed coordinate; boundedness makes its objective invariant under translation. The resulting slice is pointed. At an optimal vertex each tight-edge component connects to a fixed or zero anchor, since otherwise it could be translated locally, contradicting full active rank. A tight spanning forest therefore expresses every coordinate as an anchor plus at most $n-1$ signed demands. Its magnitude is at most $H^* + nB$ and its denominator divides k . Consequently

$$|f(p)| \leq nA_0(H^* + nB), \quad \text{den}(f(p)) \mid kq.$$

The value encoding bound is uniform and polynomial in binary input size; $\log k$ and $\log q$ are at most sums of input denominator lengths.

4 Polynomial-size domain interface and minimization

Make a node (i, t) for each $i \in I$, $l_i < t \leq h_i$, meaning $p_i \geq t$. The ground set W has size at most $2n|I|$. Encode $A(p) = \{(i, t) : l_i < t \leq p_i\}$. Union/intersection correspond to join/meet. The feasible family $\mathcal{C} = \{A(p) : p \in D\}$ has explicit implications:

1. Prefix arcs $(i, t) \rightarrow (i, t-1)$ when $t > l_i + 1$.

2. For an integer–integer edge put $d = \lceil b_{uv} \rceil$. The base condition $p_u \geq l_v + d$ is automatic if $l_v + d \leq l_u$, forces node $(u, l_v + d)$ if in range, and makes the family empty if above h_u .
3. Each node (v, t) implies $(u, t + d)$. Targets at most l_u are automatic; targets above h_u forbid (v, t) ; other targets give arcs.

These conditions are equivalent to $p \in D$, including the essential base condition when $p_v = l_v$. There are $O(n^2 + n|E|)$ arcs/nodes and polynomial-bit labels, independently of numerical denominators.

A member contains all forced nodes, avoids all forbidden nodes and is closed under arcs. Reachability computes the smallest member and the smallest member containing any given node, or detects its nonexistence. This is the ring-family interface of Schrijver [7, §6], supplied explicitly rather than assumed from a value oracle. Equivalently compute the intersection and union of members and the closure digraph on the remaining nodes, giving the lattice encoding of [8, §10.3].

Apply established exact rational submodular minimization to $F(A(p)) = f(p)$. The uniform encoding bound above and [8, §10.2, problem 10.3.8 and its reduction] give polynomial oracle-bit complexity on this lattice; [7, §6] supplies the classical combinatorial ring reduction as well. Counting arithmetic operations alone would not suffice. All oracle calls here are polynomial-bit rational LPs. A minimizing assignment and one optimizing completion give a global mixed minimizer by Lemma 2. The full algorithm is polynomial in binary length; a strongly polynomial total LP-based algorithm is not claimed.

5 The actual closed hull and facet complexity

Let $H = nB + n$, $R = \{r : r_u - r_v \geq 0\}$ and $K = \text{conv}(T \cap [-H, H]^n)$. These are mathematical objects for a size argument, not an enumeration performed by the algorithm.

For every objective bounded below on P , the objective annihilates component translation lineality. Anchor one coordinate per graph component at zero. An optimal vertex of this pointed relaxation has tight spanning trees connecting each component to its anchor. Its coordinates are sums of at most $n - 1$ signed demands and have norm at most nB . Lemma 2 puts a mixed optimum within norm H , uniformly over all bounded objectives. Thus every such objective attains its minimum on T inside K .

The bounded mixed set is a finite union of bounded rational slice polytopes, so K is a rational polytope and $K + R$ is a closed rational polyhedron. Every $r \in R$ decomposes into nonnegative multiples of integral signed threshold layers in R . For such an integral layer d and $z \in T$, $z + md \in T$ for all nonnegative integers m . Interpolation of consecutive steps puts $z + td$ in $\text{conv}(T)$ for every real $t \geq 0$. The same holds for finite convex combinations, then finite sums of layers. Hence $K + R \subset \text{conv}(T)$.

If some $z \in T$ lay outside $K + R$, strict separation would give a with $a^T z < \inf_{w \in K+R} a^T w$. The right side is finite, so a is nonnegative on R and bounded below on P . The uniform small-optimum conclusion makes its minimum on T equal its minimum on K , a contradiction. Therefore

$$\text{conv}(T) = K + R,$$

which proves closedness of the actual hull, without substituting its closure.

Slice matrices consist of directed incidence rows, unit fixings and integer box bounds, and are totally unimodular. Each slice vertex lies in $(1/k)\mathbb{Z}^n$ and has norm at most H . Thus K is generated by finitely many such vertices; R is generated by integral threshold rays of norm at most one. The hull is full-dimensional, since original cover variables contain a translated positive orthant in the convex hull of large integral vectors. Homogenize point and ray generators as $(v, 1), (r, 0)$ and

multiply point generators by k . Integer generator coordinates have magnitude at most $k(H + 1)$. Every facet has n independent homogeneous generators on it; cofactors give an integral facet inequality, including constant term, with coefficients bounded by $n![k(H + 1)]^n$. Lineality causes no problem: spanning generators may include both signs of lineality directions.

A computable generous bound on the total signed binary encoding of each facet inequality is

$$\varphi = 10(n + 1)^2[1 + \lceil \log_2(n + 1) \rceil + \lceil \log_2(k + 1) \rceil + \lceil \log_2(H + 2) \rceil].$$

It dominates the cofactor coefficient lengths and is polynomial in input encoding. No vertices, facets or denominator residues are enumerated.

6 Strong separation, scope and validation

The optimization procedure returns exact feasible optimizing points or improving integral recession directions. Points need not be vertices. Rescale an improving ray to the normalization in the strong-optimization interface of [8, Definitions 6.2.1–6.2.2]. Together with φ , this gives a well-described rational polyhedron. The exact strong optimization–separation equivalence [8, Theorem 6.4.9, pp. 179–180] therefore provides a polynomial-time strict separator or exact membership confirmation at any rational query. Undo the diagonal sign change to obtain Theorem 1. Unbounded polyhedra and boundary queries are included; no approximate membership or closure surrogate is used.

The separate subtree-hull Conjecture 2 of [1] remains unresolved here. No polynomial-size extended formulation, facet classification, arbitrary-real oracle result or full polynomial-optimizer implementation is claimed. The finite exact checkers enumerate only small cases; they do not implement LP, submodular minimization and GLS in polynomial time or prove complexity from observed runtime. Independent adversarial families checked proximity, attainment and oracle bits, and ring/hull/separation interfaces. The supplement records primary-source comparisons, exact diagnostics and limitations of the bounded priority audit. Classical ingredients and earlier formulations receive credit above; no first-ever or exhaustive novelty certificate is asserted.

AI tools were used extensively in solving, verifying, source auditing and preparing this note and supporting materials. This is an unrefereed preprint. Independent adversarial reviews are AI reviews; conventional human peer review, journal acceptance and formal proof-assistant certification are not claimed.

References

- [1] M. Conforti, *Combinatorial Mixed-Integer Programming*, Oberwolfach Reports 51/2008, 2904–2905. Primary report.
- [2] W. Cook, A. M. H. Gerards, A. Schrijver and E. Tardos, *Sensitivity theorems in integer linear programming*, *Mathematical Programming* **34** (1986), 251–264. Author copy.
- [3] Y. Takamatsu, S. Hara and K. Murota, *Continuous/Discrete Hybrid Convex Optimization and Its Optimality Criterion*, *Trans. Institute of Systems, Control and Information Engineers* **17** (2004), 409–411 (Japanese; author English translation).
- [4] K. Murota, *On steepest descent algorithms for discrete convex functions*, *SIAM J. Optimization* **14** (2003), 699–707; author report METR2002-12, §4.1.

- [5] M. Conforti, M. Di Summa, F. Eisenbrand and L. A. Wolsey, *Network Formulations of Mixed-Integer Programs*, Mathematics of Operations Research **34** (2009), 194–209; March 2008 author manuscript.
- [6] M. Conforti, L. A. Wolsey and G. Zambelli, *Projecting an extended formulation for mixed integer covers on bipartite graphs*, Mathematics of Operations Research (2010); author manuscript revised November 2009.
- [7] A. Schrijver, *A combinatorial algorithm minimizing submodular functions in strongly polynomial time*, J. Combinatorial Theory B **80** (2000), 346–355. Author manuscript, §6.
- [8] M. Grötschel, L. Lovász and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, 1988. Inspected 1988 edition.