An explicit rank-two entropy-minimizing Fano degeneration

Alec Kriebel
Independent researcher
ORCID: 0009-0001-9320-500X

16 October 2026 Version 1.0.0

Abstract

We construct a smooth Fano fivefold with a nondivisorial global entropy-minimizing valuation and a nontrivial canonical first degeneration. The example is the product of an explicit quadric blowup with the first Hirzebruch surface. Its two positive valuation weights are specified by exact moment equations; an elementary resultant argument, using the classical Hermite–Lindemann theorem, proves that their ratio is irrational. The associated graded anticanonical ring identifies the first central variety, which is singular although the original fivefold is smooth. Global optimality follows from the established optimal-degeneration and soliton theorems after checking the product filtration and normalization. The threefold factor and its optimal degeneration are due to Miao–Wang. No claim of novelty or historical firstness is made.

1 The construction and its target

Blum's discussion of canonical Fano degenerations asks for explicit examples whose first central variety differs from the original variety, and in particular for one with a nondivisorial minimizing valuation [1, pp. 888–889]. The construction below satisfies both requirements. We work over $\mathbb{C}$. By a divisorial real valuation we mean $c\,\mathrm{ord}_F$, where F is a prime divisor over the variety and $c > 0$ is real. An irrational rescaling alone therefore does not prove nondivisoriality.

Earlier explicit nontrivial first centers include Li–Li's SO_4-compactifications [3, §6]. Their second example has a two-component optimizer; we have not verified irrationality of its direction. Miao–Wang's divisorial threefold factor already appears in their January 2024 first preprint. We cite the revised edition for precise theorem numbering. The present claim is an explicit construction, not a priority certificate.

In $\mathbb{P}^4_{[u:x_0:x_1:x_2:x_3]}$, set

$$Q = \{u^2 + x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}, \qquad H = Q \cap \{u = 0\},$$
$$C = H \cap \{x_0^2 + 2x_1^2 + 3x_2^2 + 4x_3^2 = 0\}, \qquad V = \mathrm{Bl}_C\, Q.$$

Let $E \subset V$ be the strict transform of H. Let $S = \mathbb{F}_1$ be the smooth toric surface with anticanonical polygon

$$P = \{(x, y) : -1 \le y \le 1,\ -1 \le x \le y + 1\},$$

and let $D \subset S$ be the toric divisor with inward primitive normal $(0, 1)$. Put $X = V \times S$. Define $a, b > 0$ by

$$\int_{-1}^{0} s(2+s)^2 e^{-as}\, ds + \int_{0}^{2} s(2-s)^2 e^{-as}\, ds = 0, \tag{1}$$
$$\int_{-1}^{1} y(2+y) e^{-by}\, dy = 0. \tag{2}$$

Their existence and uniqueness are proved below.

Theorem 1. *The variety X is a smooth Fano fivefold. The quasi-monomial valuation v with weights (a, b) along the transverse divisors $E \times S$ and $V \times D$ is the unique global H-minimizing valuation on X. Its value group is $a\mathbb{Z} + b\mathbb{Z}$, of rational rank two, so v is nondivisorial. Its canonical first central variety is*

$$Z \times S, \qquad Z = \mathrm{Bl}_C Q_0, \qquad Q_0 = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{P}^4.$$

This central variety is singular and hence is not isomorphic to X. It already admits the required Kähler–Ricci soliton, so the second, polystable step changes no underlying variety.

We use the global valuative H-functional in the Han–Li/Blum–Liu–Xu–Zhuang convention. For finite-discrepancy valuations its expression is

$$H(v) = A_X(v) + \log \int_{\mathbb{R}} e^{-\lambda}\, d\mu_v(\lambda),$$

where μ_v is the probability-normalized asymptotic Duistermaat–Heckman measure of the raw valuation filtration on $\bigoplus_m H^0(X, -mK_X)$; see [5, Definition 4.1]. The scale of a valuation matters for this functional.

2 Geometry and the established factor degeneration

The quadric Q and its hyperplane section H are smooth. Dependent gradients for the two equations of $C \subset \mathbb{P}^3$ would give $(i+1-\lambda)x_i = 0$ for all $i = 0, \dots, 3$. Distinct coefficients permit at most one nonzero coordinate, contradicting $\sum x_i^2 = 0$. Thus C is smooth. On $H \simeq \mathbb{P}^1 \times \mathbb{P}^1$ it is a connected $(2,2)$-curve of genus one: connectedness follows from $H^1(H, \mathcal{O}_H(-2,-2)) = 0$, and adjunction gives its genus.

Write h for the pulled-back hyperplane class and F for the blowup exceptional divisor. The codimension-two blowup formula gives $-K_V = 3h - F$. The ideal $\mathcal{I}_C(2)$ is globally generated by the second quadric equation and the products of u with the five coordinates. The graph of these generators embeds the blowup into $Q \times \mathbb{P}^N$, with second hyperplane class $2h - F$. Hence $3h - F$ is ample. Also $E \simeq H$, since C is Cartier in H.

These data put V in family 2.23(a) of [2]. Miao–Wang identify the ord_E special degeneration, its central variety Z, and its canonical weight coordinate $s = k/m - 1$ [2, §3.1, Theorems 3.1–3.2]. The weight measure has density

$$\rho(s) = \begin{cases} (2+s)^2 & -1 \le s \le 0, \\ (2-s)^2 & 0 \le s \le 2. \end{cases}$$

Its mass is 5, and its positive soliton scale is exactly the a in (1). Their weighted K-polystability theorem for $(Z, a\xi_E)$ is an explicit input here. The underlying ordinary family is the blowup along $C \times \mathbb{A}^1$ of

$$\{t^2u^2 + \textstyle\sum x_i^2 = 0\} \subset \mathbb{P}^4 \times \mathbb{A}^1.$$

The center is relatively smooth of codimension two; its regular-parameter blowup charts show flatness and identify the scheme-theoretic central fiber. The specialness and associated anticanonical grading are those of the cited ord_E degeneration.

The vertex $o = [1:0:0:0:0]$ of Q_0 is disjoint from C. The blowup is an isomorphism there, so Z has an ordinary three-dimensional double point and is smooth elsewhere. Blowing up that node resolves it with discrepancy 1; in particular it is klt and Gorenstein. The same graph argument proves $-K_Z = 3h - F$ ample.

The polygon P has cyclic inward normals $(1,0), (0,1), (-1,1), (0,-1)$, with consecutive determinants 1 and all facets at distance one. Thus S is smooth Fano. Products preserve the stated Fano and klt properties. Moreover $\mathrm{Sing}(Z \times S) = \{o\} \times S$, whereas X is smooth.

3 Exact weights and irrationality

Each left side of (1)–(2) is strictly decreasing: its derivative is a negative integral of $s^2\rho(s)e^{-ts}$ or $y^2(2+y)e^{-ty}$. Its value at zero is respectively $5/12$ or $2/3$. A fixed negative subinterval dominates as $t \to +\infty$. Hence each equation has exactly one positive root.

The x-slice of P has width $2+y$ and midpoint $y/2$. Consequently (2) kills both components of the weighted barycenter with weight e^{-by}. Strict positive definiteness of the covariance Hessian of $\log \int_P e^{-\langle \xi, p\rangle} dp$ gives the unique toric parameter $\xi = (0, b)$. Wang–Zhu's theorem supplies the soliton [4]. Their analytic point-action parameter is the negative of the filtration parameter used here; see [5, Remark 2.45].

Lemma 2. *The ratio a/b is irrational.*

Proof. Elementary integration, for $t > 0$, rewrites the two moment equations as

$$(a^3 + a^2 - 2a - 6)e^{3a} + 16ae^{2a} + 4a + 6 = 0, \tag{3}$$
$$(b^2 - 2)e^{2b} + 3b^2 + 4b + 2 = 0. \tag{4}$$

Here $b^2 \neq 2$, since the remaining coefficient would be $8 + 4b \neq 0$. If b were algebraic, (4) would make e^{2b} algebraic, contrary to Hermite–Lindemann. Thus b is transcendental.

Suppose $a/b = p/q$, with positive integers p, q, and write $r = p/q$, $z = e^{b/q}$. Then

$$A(b, z) = B(b, z) = 0,$$

where

$$A(T, U) = (r^3T^3 + r^2T^2 - 2rT - 6)U^{3p} + 16rTU^{2p} + 4rT + 6,$$
$$B(T, U) = (T^2 - 2)U^{2q} + 3T^2 + 4T + 2.$$

The polynomial $R(T) = \mathrm{Res}_U(A, B) \in \mathbb{Q}[T]$ has degree at most $6(p+q)$, by separate homogeneity of the resultant. Its coefficient in that degree is

$$\mathrm{Res}_U(r^3U^{3p},\, U^{2q} + 3) = r^{6q}3^{3p} \neq 0.$$

Indeed these are exactly the highest T-degree parts of the two polynomials in the Sylvester determinant. Thus R is nonzero. The common root z implies $R(b) = 0$, contradicting the transcendence of b. □

At the generic point of $E \times D$, choose normal parameters h, w. The usual monomial valuation of weights a, b extends to $\mathbb{C}(X)$. Equivalently, choose toric characters z, w on S with $\mathrm{ord}_D(z) = 0$, $\mathrm{ord}_D(w) = 1$; its Gauss formula is

$$v\Big(\sum f_{ij} z^i w^j\Big) = \min_{i,j}\{a\ \mathrm{ord}_E(f_{ij}) + bj\}.$$

For a fixed leading pair, distinct powers of z cannot cancel over $\mathbb{C}(E)$, so multiplication of leading terms justifies the formula and its fraction-field extension. All values lie in $a\mathbb{Z} + b\mathbb{Z}$; the parameters h, w attain a, b, giving equality. Lemma 2 makes its rational rank two. Every $c\ \mathrm{ord}_F$ has rational rank one. The valuation has ordered rank one as a subgroup of $\mathbb{R}$; it is the *rational* rank that proves nondivisoriality.

4 The real filtration and its normalization

Let $A_m = H^0(V, -mK_V)$ and $B_m = H^0(S, -mK_S)$. Künneth gives $R(X)_m = A_m \otimes B_m$. For a first-factor section of E-order k and a toric character $(i, j) \in mP \cap \mathbb{Z}^2$, the raw weight is

$$ak + b(j + m).$$

An ord_E-adapted basis tensored with character lines is adapted to v. Irrationality separates distinct (k, j); different i remain linearly independent in the residue field. Leading coefficients multiply without an additional cancellation. Therefore, as degree-graded algebras,

$$\mathrm{gr}_v R(X) \simeq (\mathrm{gr}_{\mathrm{ord}_E} R(V)) \# R(S),$$

where $\#$ is the equal-degree Segre product. The factor degeneration identifies $\mathrm{gr}_{\mathrm{ord}_E} R(V)$ with the anticanonical section ring of Z, after a common Veronese if needed. The Segre algebra is finitely generated and its Proj is $Z \times S$ with anticanonical polarization.

This is finite generation in the geometric real-filtration sense: finitely many homogeneous section generators with real weights. It is not a claim that the positive cone of the dense group $a\mathbb{Z} + b\mathbb{Z}$ is a finitely generated semigroup [7, Definition 2.2]. The ordinary E family and the toric action on S, or an adapted finite anticanonical embedding with these real diagonal weights, realize the special real degeneration with this central ring [5, §2.2, Geometric R-TC I]. Rational directions approximate it by ordinary special degenerations; no such rational direction is asserted to minimize H.

Both transverse divisors have discrepancy one on the smooth X, so $A_X(v) = a + b$. The canonical anticanonical weight is consequently

$$ak + b(j + m) - m(a + b) = a(k - m) + bj. \tag{*}$$

Thus the induced central parameter is precisely $(a\xi_E, b(0, 1))$. In particular no unaccounted character twist or sign change appears when applying the optimality criterion.

5 Global optimality and the canonical first step

We spell out the analytic product step to avoid assuming a tensorization theorem for H-minimizers. The polystability of $(Z, a\xi_E)$ from [2], together with [7, 6], gives its weak soliton. The toric factor has the soliton specified above. Use the Han–Li convention

$$e^{-\theta_\phi(\xi)} \mathrm{MA}_n(\phi) = e^{-\phi}, \qquad \mathrm{Ric}(\omega) - \omega = -dd^c\theta_\phi(\xi), \quad \mathrm{MA}_n(\phi) = \omega^n/n!,$$

with harmless constant normalizations of the canonical volume measure [5, (110)–(111), Remark 2.45]. For factor potentials ϕ_Z, ϕ_S, put $\phi = \phi_Z + \phi_S$, $\theta = \theta_Z + \theta_S$. The binomial expansion gives exactly

$$\mathrm{MA}_5(\phi) = \mathrm{MA}_3(\phi_Z) \otimes \mathrm{MA}_2(\phi_S).$$

The factor equations multiply to the same equation on $Z \times S$ for the sum parameter. The potentials are bounded on a resolution [6, Corollary 31, Proposition 32]; the product Monge–Ampère measure and klt canonical measure have no mass on the analytic singular set. Hence the equality extends from the regular locus as a weak equation. This also follows in the finite-energy formulation of [6].

By [7, Theorem 1.3], the central pair is weighted K-polystable, in particular semistable. Han–Li's criterion [5, Theorem 5.3], with their comparison of infima [5, Theorem 1.5], now proves that v minimizes H among *all* valuations on the original X. The special real degeneration, its klt anticanonical central ring, finite discrepancy, and exact induced parameter were checked above; no restriction to product-form competitors is made. Blum–Liu–Xu–Zhuang's absolute uniqueness theorem [7, Theorem 1.2] identifies this valuation and $\mathrm{Proj}\,\mathrm{gr}_v R(X)$ with the canonical first step. The central pair is already polystable, so its second step changes no underlying variety. The smooth/singular distinction and Lemma 2 complete Theorem 1.

6 Scope and verification

This is an explicit construction from the cited general theorems and Miao–Wang's established threefold example. We do not claim historical firstness, a new general optimal-degeneration theorem, a three-dimensional nondivisorial example, or an exhaustive priority determination. The statement resolves the specified construction request in dimension five without a product exclusion. The accompanying archive preserves the original proof and corrections, primary-source fingerprints, independent adversarial analyses, and exact auxiliary checks. Numerical brackets and finitely many resultants do not prove irrationality; the universal argument above does. Successful software checks do not formally verify the geometric or analytic theorem inputs.

AI tools were used extensively in developing, checking, and preparing this work. The reported adversarial reviews were conducted by AI agents. This preprint is unrefereed; independent human peer review and formal verification have not been completed.

References

[1] H. Blum, Degenerations of unstable Fano varieties and Kahler–Ricci soliton, in *Toric Geometry*, *Oberwolfach Rep.* 19 (2022), no. 2, Report 17/2022, pp. 888–889; published online 14 April 2023. doi:10.4171/OWR/2022/17.

[2] M. Miao and L. Wang, *Optimal Degenerations of K-unstable Fano threefolds*, first posted 25 January 2024; cited edition arXiv:2401.13999v2, 13 April 2026. arXiv:2401.13999v2.

[3] Y. Li and Z. Li, Equivariant $\mathbb{R}$-test configurations and semistable limits of $\mathbb{Q}$-Fano group compactifications, *Peking Math. J.* 6 (2023), 559–607. doi:10.1007/s42543-022-00054-0. The inspected example text is arXiv:2103.06439v6.

[4] X.-J. Wang and X. Zhu, Kähler–Ricci solitons on toric manifolds with positive first Chern class, *Adv. Math.* 188 (2004), 87–103. doi:10.1016/j.aim.2003.09.009.

[5] J. Han and C. Li, Algebraic uniqueness of Kähler–Ricci flow limits and optimal degenerations of Fano varieties, *Geom. Topol.* 28 (2024), 539–592. The cited theorem numbering is that of arXiv:2009.01010v1.

[6] J. Han and C. Li, On the Yau–Tian–Donaldson conjecture for generalized Kähler–Ricci soliton equations, *Comm. Pure Appl. Math.* 76 (2023), 1793–1867. doi:10.1002/cpa.22053. The cited corollary/proposition locators refer to the inspected author manuscript.

[7] H. Blum, Y. Liu, C. Xu and Z. Zhuang, The existence of the Kähler–Ricci soliton degeneration, *Forum Math. Pi* 11 (2023), e9. doi:10.1017/fmp.2023.5. The inspected preprint is arXiv:2103.15278v2.