

Quadratic convergence of SOS-density bounds on convex bodies

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Abstract

For every fixed compact convex body $K \subset \mathbb{R}^n$ with nonempty interior and every real polynomial f , the normalized Lebesgue sum-of-squares density bound of degree at most $2r$ has error $O_{K,f}(r^{-2})$ for every integer $r \geq 1$. We give a self-contained proof and an explicit constant, using one global square. A quadratic Taylor upper estimator reduces the objective to a scalar endpoint problem. Homothetic contraction compares every sublevel shell with the same core mass that normalizes a fixed power of a Dirichlet polynomial; that mass cancels before summation. This proves the quadratic upper rate in the convex-body/Lebesgue regime of the 2018 Oberwolfach question. A classical interval example shows that the exponent is sharp in the worst case. The proxy and the broad localization and relative-mass mechanisms have established precedents; no firstness or new-method claim is made.

1 Statement and source context

All integrals over K use n -dimensional Lebesgue measure. Let Σ_r be the set of polynomials that are sums of squares of real polynomials of degree at most r . These are exactly the global sum-of-squares polynomials of degree at most $2r$: the highest-degree homogeneous squares cannot cancel. For a compact set K of positive volume and a real polynomial f , define

$$H_r(f, K) = \inf \left\{ \int_K f(x)q(x) dx : q \in \Sigma_r, \quad \int_K q(x) dx = 1 \right\}. \quad (1)$$

The constant density makes this set nonempty, and positivity gives $H_r(f, K) \geq f_* := \min_K f$.

Theorem 1. *Let $n \geq 1$, let $K \subset \mathbb{R}^n$ be compact and convex with nonempty interior, and let $f \in \mathbb{R}[x_1, \dots, x_n]$. Choose any $a \in K$ with $f(a) = f_*$. Set*

$$\begin{aligned} H &= \max_{x \in K} \|\nabla^2 f(x)\|_{\text{op}}, & \gamma &= 1 + H/2, \\ g(x) &= \nabla f(a) \cdot (x - a) + \gamma \|x - a\|^2, & G &= \max_{x \in K} g(x), \\ L &= n + 2, & C_n &= 4L^2 \left(\frac{\pi}{2}\right)^{2L} (1 + 2^{n+2}). \end{aligned}$$

Then $0 < G < \infty$ and, for every integer $r \geq 1$,

$$0 \leq H_r(f, K) - f_* \leq \frac{C_n G}{r^2}. \quad (2)$$

A single global square, normalized on the whole of K , witnesses the upper bound.

No smoothness or semialgebraicity of the boundary is assumed, and f need not be convex. The constant is deliberately coarse. If $D = \text{diam}(K)$ and $B = \max_K \|\nabla f\|$, then $G \leq BD + (1 + H/2)D^2$, so a bound independent of the choice of minimizer is available. The construction is an existence proof; it does not give an efficient method for locating a .

Laurent's contribution, joint work with de Klerk, to the 2018 Oberwolfach report asks for the convergence regime for general convex bodies with Lebesgue measure [1]. Its displayed density constraints (3)–(4) omit the value of the integral. We explicitly use unit mass, as required by its probability formulation and by the full hierarchy definition (1) in Slot–Laurent [2, Eq. (3)]. Thus our statement concerns the normalized problem, rather than a literal unnormalized infimum. Slot–Laurent's Theorem 11 gives $O(\log^2 r/r^2)$ for this convex-body/Lebesgue hierarchy, and their concluding questions ask whether the logarithm can be removed. Theorem 1 supplies the quadratic upper rate in that precise setting.

The constrained-gradient and quadratic-upper-estimator reductions are already present in de Klerk–Laurent [3, preprint Cor. 4.3] and Slot–Laurent [2, Lemmas 1 and 4]. Endpoint polynomial concentration is classical. In particular, Totik–Varga [4, Thm. 2.1], with endpoint parameter $a = 0$ and $\psi(v) = \sqrt{v}$, supplies degree- m polynomials equal to 1 at zero, bounded by 2 on $[0, 1]$, and bounded by $A \exp[-c(m^2 t)^{1/4}]$ for $t > 0$, with $A, c > 0$ independent of m . Markov's inequality gives a positive core of width proportional to m^{-2} ; the relative-mass estimate below then gives another deduction of our theorem from those older ingredients. Totik–Varga also compare shell masses with local core mass in their Christoffel-function arguments. We assign no originality to the proxy, endpoint localization, or broad shell/core cancellation mechanism.

Our proof instead uses an explicit finite trigonometric identity and imports no approximation-theory convergence theorem. The accompanying source audit distinguishes a deduction assembled here from an earlier explicitly printed resolution. Its bounded search found no such full-target theorem in the inspected passages; this does not establish firstness, historical disclosure, or current worldwide openness. The weaker rate recorded in the final 2026 overview [5, Table 1] is evidence about that source's contents, not an openness certificate.

2 An elementary concentration proof

Quadratic proxy and relative sublevel mass

For $x \in K$, convexity places $a + s(x - a)$ in K for $0 \leq s \leq 1$. The right derivative at zero of $f(a + s(x - a))$ is nonnegative, so

$$\ell(x) := \nabla f(a) \cdot (x - a) \geq 0.$$

Taylor's formula on this segment gives

$$0 \leq f(x) - f(a) \leq \ell(x) + \frac{H}{2} \|x - a\|^2 \leq g(x).$$

The added positive quadratic ensures $g(x) > 0$ for $x \in K \setminus \{a\}$. Full dimension and compactness therefore give $0 < G < \infty$. The polynomial $u = g/G$ has degree at most two and takes values in $[0, 1]$ on K . For $0 \leq \alpha \leq 1$,

$$g(a + \alpha(x - a)) = \alpha \ell(x) + \alpha^2 \gamma \|x - a\|^2 \leq \alpha g(x). \quad (3)$$

For every $t > 0$ put $S_t = \{x \in K : u(x) \leq t\}$ and $F(t) = \text{vol}_n(S_t)$, so that $F(t) = \text{vol}_n(K)$ for $t \geq 1$. If $0 < s \leq t$, the homothety $x \mapsto a + (s/t)(x - a)$ maps S_t into S_s by (3). Its Jacobian is $(s/t)^n$,

hence

$$F(t) \leq (t/s)^n F(s) \quad (0 < s \leq t). \quad (4)$$

In particular $F(h) \geq h^n \text{vol}_n(K) > 0$ for $0 < h \leq 1$. The estimate also holds when $t > 1$, as needed for an outer shell. No density formula or differentiation of F is used.

A polynomial endpoint kernel

For an integer $m \geq 1$, let $N = 2m + 1$ and define

$$A_m(t) = \frac{1 + 2 \sum_{j=1}^m T_j(1 - 2t)}{N}, \quad (5)$$

where $T_j(\cos \theta) = \cos(j\theta)$ is the Chebyshev polynomial. Thus A_m is a real polynomial of degree m and $A_m(0) = 1$. Writing $v = \arcsin \sqrt{t}$ for $0 < t \leq 1$, the finite Dirichlet-sum identity gives

$$A_m(t) = \frac{\sin(Nv)}{N \sin v} = \frac{\sin(N \arcsin \sqrt{t})}{N \sqrt{t}}. \quad (6)$$

The defining sum and the sine quotient respectively imply

$$|A_m(t)| \leq \min\{1, (N\sqrt{t})^{-1}\}, \quad 0 < t \leq 1. \quad (7)$$

Put $h = N^{-2}$. Convexity of arcsine gives $\arcsin z \leq \pi z/2$ on $[0, 1]$. For $0 < t \leq h$ we consequently have $Nv \leq \pi/2$. Using $\sin w \geq 2w/\pi$ there, and $v \geq \sqrt{t}$, in (6) yields

$$A_m(t) \geq 2/\pi \quad (0 \leq t \leq h), \quad (8)$$

including $t = 0$ since $A_m(0) = 1$.

Cancellation of the core mass

Take $L = n + 2$ and $Q_m(x) = A_m(u(x))^{2L}$. This is the global square $[A_m(u(x))]^L$, regardless of its values off K . Equations (7)–(8) give

$$Z_m := \int_K Q_m \, dx \geq (2/\pi)^{2L} F(h) > 0, \quad Q_m \leq \min\{1, (h/u)^L\} \quad (u > 0). \quad (9)$$

The core $\{u \leq h\}$ contributes at most $hF(h)$ to $\int_K u Q_m \, dx$. For the pairwise disjoint shells

$$E_j = \{x \in K : 2^j h < u(x) \leq 2^{j+1} h\}, \quad j = 0, 1, \dots,$$

equation (4) gives

$$\begin{aligned} \int_{E_j} u Q_m \, dx &\leq 2^{j+1} h \, 2^{-jL} F(2^{j+1} h) \\ &\leq h F(h) \, 2^{n+1} 2^{j(n+1-L)}. \end{aligned}$$

Only finitely many shells meet K . Extending their nonnegative bounds to an infinite sum is legitimate, and $L = n + 2$ gives $\sum_{j \geq 0} 2^{j(n+1-L)} = 2$. Thus

$$\int_K u Q_m \, dx \leq h F(h) (1 + 2^{n+2}).$$

Dividing by (9) cancels the same $F(h)$:

$$\frac{\int_K u Q_m \, dx}{Z_m} \leq \left(\frac{\pi}{2}\right)^{2L} \frac{1 + 2^{n+2}}{(2m + 1)^2}. \quad (10)$$

Zeros and oscillation of A_m away from the core cause no difficulty; the denominator uses its positive core and the tails use absolute upper bounds over the whole body.

Degree and every integer order

For $r \geq 2L$ choose $m = \lfloor r/(2L) \rfloor \geq 1$. Since $\deg u \leq 2$,

$$\deg(A_m(u)^L) \leq 2mL \leq r, \quad \deg Q_m \leq 4mL \leq 2r.$$

The density

$$q_r(x) = \frac{Q_m(x)}{Z_m} = \left[\frac{A_m(u(x))^L}{\sqrt{Z_m}} \right]^2$$

is therefore feasible for (1). As $f - f(a) \leq Gu$, (10) bounds its error. Also $2\lfloor r/(2L) \rfloor + 1 \geq r/(2L)$, giving (2) with the stated C_n . For $1 \leq r < 2L$, use $q_r = 1/\text{vol}_n(K)$, whose error is at most G . Here $C_n/r^2 \geq 1$, so the same bound holds. This proves Theorem 1 for every integer $r \geq 1$. $\square$

3 The known worst-case exponent

Interval linear objectives already give matching quadratic lower rates in de Klerk–Laurent [6, preprint Thm. 3.1 and Cors. 3.2–3.3]. The following elementary, coarse bound recalls why the universal exponent cannot be improved.

Proposition 1. *For $K = [0, 1]$ and $f(t) = t$,*

$$H_r(f, K) \geq \frac{1}{4(r+1)^2} \quad (r \geq 1).$$

The same lower bound holds for $K = [0, 1] \times [-1, 1]^{n-1}$ and $f(t, y) = t$ in every fixed dimension $n \geq 1$.

Proof. Let P_j be the Legendre polynomial normalized by $P_j(1) = 1$. The classical identities

$$\int_{-1}^1 P_j(x) P_k(x) dx = \frac{2\delta_{jk}}{2j+1}, \quad |P_j(x)| \leq 1 \quad (-1 \leq x \leq 1)$$

can be obtained directly from Rodrigues' formula $P_j(x) = (2^j j!)^{-1} (d/dx)^j (x^2 - 1)^j$: integration by parts gives orthogonality and the norm. For the supremum bound, expand the finite-polynomial identity

$$P_j(x) = \frac{1}{\pi} \int_0^\pi (x + \iota \sqrt{1-x^2} \cos \theta)^j d\theta, \quad \iota^2 = -1;$$

the base has modulus at most 1. Consequently $\ell_j(t) = \sqrt{2j+1} P_j(2t-1)$ is an orthonormal basis on $[0, 1]$ for polynomials of degree at most r when $0 \leq j \leq r$. Cauchy–Schwarz gives, for each such polynomial p ,

$$|p(t)|^2 \leq \left(\sum_{j=0}^r \ell_j(t)^2 \right) \int_0^1 p(s)^2 ds \leq (r+1)^2 \int_0^1 p(s)^2 ds.$$

If $q = \sum_j p_j^2$ is feasible, summing shows $0 \leq q(t) \leq M$, where $M = (r+1)^2$, since its integral is 1. With $\alpha = 1/(2M)$, we have $\int_0^\alpha q \leq 1/2$; hence

$$\int_0^1 tq(t) dt \geq \alpha \int_\alpha^1 q(t) dt \geq \frac{1}{4M}.$$

Taking the infimum proves the interval assertion. On the product body, apply the polynomial bound with y fixed to every $p_j(t, y)$, then sum and integrate over y . The resulting marginal density in t has integral 1 and is bounded by M , so the same argument applies. $\square$

Together with Theorem 1, this gives one fixed instance with error $\Theta(r^{-2})$ in each dimension. It does not give a positive matching lower rate for each objective or each body; constant objectives have zero error.

4 Scope, supporting material, and review status

The reference measure in Theorem 1 is Lebesgue measure. The homothety identity is not asserted for arbitrary reference measures. The construction also uses a quadratic proxy g , rather than densities constrained to be polynomials of the original objective f ; it therefore does not settle the same rate for that more restrictive hierarchy for general nonconvex f . The proof extends to f of class C^2 on an open neighborhood of K , since the same Taylor bound applies and the density still uses only the quadratic proxy.

The accompanying verification package contains proof and source audits, independent derivations and adversarial reviews, exact arithmetic consistency checks, and an exact moment-matrix formulation at the nonsimple apex of a square pyramid. These computations and finite checks supplement the analytic proof; they do not certify its universal quantifiers. Source retrieval, passages actually read, and bounded literature-search limitations are recorded separately.

AI tools were used extensively in developing the solution, derivations, source checks, verification, drafting, and adversarial reviews. Independent reviewing agents were AI systems, not human peer reviewers. This research note is unrefereed; it has not received external human peer review and is not a proof-assistant or other machine-formalized proof certificate. The author is responsible for its mathematical claims and presentation.

References

- [1] M. Laurent (joint work with E. de Klerk), Convergence analysis of Lasserre measure-based approximations for polynomial optimization, in *Combinatorial Optimization*, Oberwolfach Reports **15**(4), report 50/2018 (2018), 2991–2993. doi:10.4171/owr/2018/50.
- [2] L. Slot and M. Laurent, Improved convergence analysis of Lasserre’s measure-based upper bounds for polynomial minimization on compact sets, *Mathematical Programming* **193** (2022), 831–871. arXiv:1905.08142v2 (2020). doi:10.1007/s10107-020-01468-3. Theorem and lemma numbers cited here refer to this preprint version.
- [3] E. de Klerk and M. Laurent, Comparison of Lasserre’s measure-based bounds for polynomial optimization to bounds obtained by simulated annealing, *Mathematics of Operations Research* **43**(4) (2018), 1317–1325. doi:10.1287/moor.2017.0906. arXiv:1703.00744v1.
- [4] V. Totik and T. Varga, Non-symmetric fast decreasing polynomials and applications, *Journal of Mathematical Analysis and Applications* **394** (2012), 378–390. doi:10.1016/j.jmaa.2012.03.006.
- [5] M. Laurent and L. Slot, An overview of convergence rates for sum of squares hierarchies in polynomial optimization, in *Recent Developments in Industrial and Applied Mathematics*, ICIAM2023 Springer Series **1** (2026), 149–176. doi:10.1007/978-981-95-1446-5_8. Author-hosted final text.
- [6] E. de Klerk and M. Laurent, Worst-case examples for Lasserre’s measure-based hierarchy for polynomial optimization on the hypercube, *Mathematics of Operations Research* **45**(1) (2020), 86–98. doi:10.1287/moor.2018.0983. arXiv:1804.05524v1.