

Pure point Dirichlet spectrum on quasi-conical domains

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Abstract

For every dimension at least two, we construct a connected open quasi-conical domain whose Dirichlet Laplacian has a complete orthonormal eigenbasis, although its spectrum and essential spectrum are the entire nonnegative half-line. Its positive eigenvalues are dense, zero is not an eigenvalue, and both continuous spectral subspaces vanish. We use the expanding-cube and shrinking-window geometry of Krejčířík and Lotoreichik, choosing both cube sizes and positive apertures inductively. The additional construction transports a growing family of complete finite spectral clusters with summable norm errors. A repeated dense test family proves completeness and excludes singular continuous spectrum. This answers the existential spectral-type question in their discussion and the corresponding Oberwolfach contribution.

Status and disclosure. AI tools were used extensively in developing, reproducing, checking and adversarially reviewing the argument and preparing this note. This is an unrefereed preprint. Internal AI-assisted review is not external human peer review or proof-assistant certification. The source and verification package records the checks and their limits.

1 The question and the precise result

An open set is *quasi-conical* if it contains Euclidean balls of arbitrarily large radius. On any open $\Omega \subset \mathbb{R}^d$, let A_Ω be the nonnegative self-adjoint Dirichlet Laplacian defined by the closed form

$$q_\Omega[u] = \int_\Omega |\nabla u|^2, \quad \text{dom } q_\Omega = H_0^1(\Omega).$$

This definition imposes no regularity assumption on the boundary. Krejčířík and Lotoreichik [1] constructed connected quasi-conical domains with an embedded eigenvalue and with no absolutely continuous spectrum. Their Section 5 asks whether the singular continuous part can also be removed. Lotoreichik's Oberwolfach contribution [2, pp. 2077–2078] poses the same existential question; it is repeated in [3, Open Problem 1, PDF p. 16].

Theorem 1. *For every integer $d \geq 2$ there is a connected open quasi-conical set $\Omega \subset \mathbb{R}^d$, consisting of expanding cubes joined through positive shrinking windows in their common faces, such that $A = A_\Omega$ has a complete orthonormal eigenbasis and*

$$\sigma(A) = \sigma_{\text{ess}}(A) = \overline{\sigma_p(A)} = [0, \infty), \quad \ker A = \{0\}, \quad \mathcal{H}_{\text{ac}}(A) = \mathcal{H}_{\text{sc}}(A) = \{0\}.$$

Both the cube widths and the window widths are chosen in the construction.

Pure point spectral type means completeness of the eigenvectors. It does not assert that every point of $[0, \infty)$ is an eigenvalue. The theorem concerns zero-thickness interface screens with open windows. It makes no assertion for arbitrary preassigned cube sizes, smooth boundaries, Neumann conditions or positive-length connecting passages; no effective numerical aperture bounds are supplied.

The geometry, freedom to choose future cube sizes and finite-window continuity below come from [1, Sections 2.2 and 3]. Whole spectral families already handle multiplicities in [4, Proposition 1.4]; see also the explicit warning in [1, Remark 3.1]. The additional step here is the exhaustive geometric transport and its completeness estimate. Neither finite-cluster perturbation theory nor diagonalization on finite reducing subspaces is claimed as a new abstract principle.

2 A bounded shrinking-window step

Write $Q_a(x) = xe_1 + (-a, a)^d$. Suppose that a bounded connected partial tower O_n ends in $Q_{a_n}(x_n)$, with all its earlier windows fixed. For $a > a_n + 1$, put $x = x_n + a_n + a$, $p = (x_n + a_n)e_1$, and

$$D_0 = O_n \cup Q_a(x), \quad D_\delta = D_0 \cup (p + (-\delta, \delta)^d), \quad 0 < \delta < a_n.$$

The connector away from its middle face is already inside the adjacent cubes. Thus $D_\delta \setminus D_0$ is precisely a patch of that face: the L^2 spaces are canonically the same, while the form domains differ. The full connector cube makes D_δ open and connected.

Lemma 2. *On this common L^2 space,*

$$\|(A_{D_\delta} + 1)^{-1} - (A_{D_0} + 1)^{-1}\| \longrightarrow 0 \quad (\delta \downarrow 0).$$

All these bounded-domain Dirichlet operators have compact resolvent and strictly positive eigenvalues.

Proof. This is the finite-window application of [1, Propositions 2.2–2.3 and Section 3]. We give a direct form argument to make the dimension and boundary hypotheses explicit. Choose one bounded box B containing the fixed finite geometry. Zero extension realizes $V_\delta = H_0^1(D_\delta)$ and $V_0 = H_0^1(D_0)$ as closed subspaces of $H_0^1(B)$. As $\delta \downarrow 0$ these spaces decrease, and

$$\bigcap_{0 < \delta < a_n} V_\delta = V_0. \tag{1}$$

Indeed, for bounded u in this intersection choose a smooth cutoff ζ_r that vanishes near p and is one outside a shrinking neighborhood. Choose δ so small that the connector is inside the zero neighborhood. Multiplying smooth compactly supported approximants to u in D_δ by ζ_r shows that $\zeta_r u \in V_0$. For $d \geq 3$ use a cutoff between radii r and $2r$; its squared gradient norm is $O(r^{d-2})$. For $d = 2$, choose a smooth θ equal to zero on $(-\infty, 0]$ and one on $[1, \infty)$, and use

$$\zeta_r(y) = \theta\left(\frac{\log(|y - p|/r^2)}{\log(1/r)}\right) \quad (r^2 < |y - p| < r),$$

continued by zero inside and one outside. Its gradient energy is

$$\int_{\mathbb{R}^2} |\nabla \zeta_r|^2 = \frac{2\pi}{\log(1/r)} \int_0^1 |\theta'(s)|^2 ds \longrightarrow 0.$$

In either case $\zeta_r u \rightarrow u$ in $H^1(B)$ for bounded u : use dominated convergence and $\|u \nabla \zeta_r\|_2 \leq \|u\|_\infty \|\nabla \zeta_r\|_2$. Truncate the real and imaginary parts first to treat arbitrary u ; Sobolev truncation preserves each V_δ and converges in H^1 . Closedness proves (1).

If $\delta_j \rightarrow 0$ and $u_j \in V_{\delta_j}$ converges weakly in $H^1(B)$, its limit belongs to every fixed V_s , hence to V_0 . Every $v \in V_0$ is its own recovery sequence. Now write $R_\delta = (A_{D_\delta} + 1)^{-1}$ and $R_0 = (A_{D_0} + 1)^{-1}$. For bounded inputs f_j , the solutions $R_{\delta_j} f_j$ are uniformly bounded in $H_0^1(B)$. If norm convergence failed, select unit inputs with a positive lower bound on $\|(R_{\delta_j} - R_0)f_j\|_2$. Pass to $f_j \rightharpoonup f$ and, by Rellich compactness, to outputs converging strongly in $L^2(B)$ and weakly in $H_0^1(B)$. The preceding form-space limit puts their limit in V_0 . Testing the variational equation against V_0 identifies it as $R_0 f$. Compactness of R_0 also gives $R_0 f_j \rightarrow R_0 f$, a contradiction. The same containing-box embedding proves compact resolvents, and the Poincaré inequality in B proves positivity. $\square$

This argument keeps every old window fixed. In dimension one a positive connector joins adjacent intervals for every $\delta > 0$, and the point removal argument fails; this is why $d \geq 2$ is required.

Suppose S is a finite set of old eigenvalues and the new cube has no eigenvalue in S . Each $\lambda \in S$ is then isolated in A_{D_0} with exactly its old full eigenspace, extended by zero. Choose disjoint isolating intervals I_λ with endpoints outside $\sigma(A_{D_0})$. Lemma 2 implies

$$E_{D_\delta}(I_\lambda) \longrightarrow E_{O_n}(\{\lambda\}) \oplus 0 \quad \text{in operator norm.} \quad (2)$$

For example, apply fixed Riesz contours around the isolated nonzero eigenvalues $(1 + \lambda)^{-1}$ of the bounded resolvents. The resolvent identity gives convergence of the contour integrals. Projections at norm distance below one have equal ranks: either projection is injective on the other's range. A multiple eigenvalue may split inside its interval; its entire cluster is retained.

3 Inductive protection of an exhaustive family

Set $a_1 = x_1 = 1$ and $O_1 = Q_1(1)$. Once O_n is fixed, the next cube eigenvalues are

$$\frac{\pi^2(m_1^2 + \cdots + m_d^2)}{4a^2}, \quad m_i \in \mathbb{N}, \quad \mathbb{N} = \{1, 2, \dots\}.$$

For any finite positive set S and $a \in (a_n + 1, a_n + 2)$, an equality with $\lambda \in S$ bounds every m_i . Only finitely many widths are forbidden. We may therefore choose a_{n+1} in this interval avoiding S , then set $x_{n+1} = x_n + a_n + a_{n+1}$. This changes no earlier parameter.

When cube $\omega_j = Q_{a_j}(x_j)$ is chosen, fix its orthonormal sine basis $(f_{j,\ell})_{\ell \geq 1}$, extended by zero as necessary. Let

$$\varepsilon_n = 2^{-n-2}, \quad \eta_n = 2^{-n}, \quad F_n = \{f_{j,\ell} : j, \ell \leq n\}.$$

We construct finite-rank orthogonal projections P_n^k , $1 \leq k \leq n$, each a sum of complete eigenspace projections of A_{O_n} , such that

$$P_n^1 \leq \cdots \leq P_n^n, \quad (3)$$

$$\|P_{n+1}^k - P_n^k\| < \varepsilon_n, \quad \text{rank } P_{n+1}^k = \text{rank } P_n^k \quad (k \leq n), \quad (4)$$

$$\|(I - P_n^n)f\| < \eta_n \quad (f \in F_n). \quad (5)$$

In (4) the old projections are extended by zero into the new finite-stage L^2 space.

For $n = 1$, a sufficiently high finite spectral cutoff gives (5). At stage n , let S_n be the finite set of eigenvalues occurring in P_n^n . Choose the next cube nonresonant with S_n . For each $\lambda \in S_n$ take an interval as in (2). If $S_n^k \subset S_n$ defines P_n^k , transport it by

$$P_{n+1}^k = \sum_{\lambda \in S_n^k} E_{D_{\delta_n}}(I_\lambda).$$

There are finitely many constraints. Choose one positive δ_n so small that (4) holds for every $k \leq n$ and

$$\delta_n < \min(a_n/2, 2^{-n}).$$

Set $O_{n+1} = D_{\delta_n}$. Disjoint intervals and complete clusters preserve nesting, complete-eigenspace status and rank. Finally choose a spectral cutoff P_{n+1}^{n+1} above all transported clusters and high enough to satisfy (5) for F_{n+1} . Compact-resolvent completeness permits this finite choice. An older projection need not remain a low-energy cutoff after transport. No simple spectrum or uniform spectral-gap estimate is assumed.

4 The final domain and its resolvent

Let $\Omega = \bigcup_n O_n$. It is open and connected and contains balls of radius $a_n \rightarrow \infty$. Apart from countably many null interface patches it is the disjoint union of the cubes. Hence

$$H = L^2(\Omega) = \bigoplus_{j \geq 1} L^2(\omega_j),$$

and all $f_{j,\ell}$ together form an orthonormal basis of H . Every earlier projection and resolvent now acts by zero extension on H . The preceding norms and ranks are unchanged by these canonical embeddings.

Put $R = (A_\Omega + 1)^{-1}$ and $R_n = i_n(A_{O_n} + 1)^{-1}r_n$, where r_n restricts and i_n extends by zero. Then $R_n \rightarrow R$ strongly, with $\|R_n\| \leq 1$. Indeed the zero-extended $V_n = H_0^1(O_n)$ are increasing closed subspaces of $V = H_0^1(\Omega)$ for

$$b(u, v) = \int_\Omega (\nabla u \cdot \overline{\nabla v} + u \bar{v}).$$

Their union is dense: compact support in the increasing open cover lies in one finite stage, and $C_c^\infty(\Omega)$ is form-dense. For $f \in H$, $R_n f$ is the b -orthogonal projection of Rf onto V_n . It therefore converges in form norm and in H . Norm convergence of this infinite exhaustion is not required. The extended R_n have kernels on the future cube summands; only R is used as a genuine resolvent on all of H .

5 Limiting eigenvectors and completeness

For fixed k , (4) gives a norm limit P^k and

$$\|P^k - P_n^k\| \leq \sum_{j=n}^{\infty} \varepsilon_j = 2^{-n-1} \quad (n \geq k). \quad (6)$$

It is an orthogonal projection of the same finite rank, by norm continuity and the distance-below-one criterion. Nesting passes to the limits. Since $R_n P_n^k = P_n^k R_n$, strong convergence of the uniformly

bounded R_n and norm convergence of P_n^k give $RP^k = P^kR$. Thus $\text{Ran } P^k$ is a finite-dimensional reducing subspace for R . Positivity and injectivity of R make every eigenvalue μ on that space strictly positive. If $Rv = \mu v$, then $v = \mu^{-1}Rv \in \text{dom } A_\Omega$ and $A_\Omega v = (\mu^{-1} - 1)v$. Consequently $\text{Ran } P^k \subset \mathcal{H}_{\text{pp}}(A_\Omega)$, the closed span of its eigenvectors. Injectivity prevents a nonzero protected limit space from escaping to infinite energy.

Fix $f = f_{j,\ell}$. For every $n \geq \max(j, \ell)$, (5) and (6), applied to the trajectory born at n , give

$$\|(I - P^n)f\| \leq \|(I - P_n^n)f\| + \|P_n^n - P^n\| < 2^{-n} + 2^{-n-1} \rightarrow 0.$$

Every $P^n f$ lies in the closed space $\mathcal{H}_{\text{pp}}(A_\Omega)$. It follows that every cube-basis probe lies there and $\mathcal{H}_{\text{pp}}(A_\Omega) = H$. This proves absence of both continuous spectral subspaces; dense eigenvalues alone would not prove that assertion. Equivalently, $P^k \rightarrow I$ strongly. Diagonalizing R on $\text{Ran } P^1$ and the finite-dimensional orthogonal differences $\text{Ran}(P^{k+1} - P^k)$ gives a complete orthonormal eigenbasis.

6 Spectral support and the zero endpoint

Nonnegativity gives $\sigma(A_\Omega) \subset [0, \infty)$. For $\lambda \geq 0$, choose $\xi \in \mathbb{R}^d$ with $|\xi|^2 = \lambda$ and $\chi \in C_c^\infty(B_1(0))$ of L^2 norm one. With cube center c_n and $r_n = a_n/2$, put

$$u_n(x) = r_n^{-d/2} \chi((x - c_n)/r_n) e^{i\xi \cdot x}.$$

These unit functions belong to $C_c^\infty(\Omega) \subset \text{dom } A_\Omega$. The product rule gives

$$\|(A_\Omega - \lambda)u_n\|_2 \leq r_n^{-2} \|\Delta \chi\|_2 + 2|\xi| r_n^{-1} \|\nabla \chi\|_2 \rightarrow 0.$$

Their supports lie in disjoint cubes, so they are orthonormal and weakly null. Weyl's criterion puts every $\lambda \geq 0$ in the essential spectrum. Completeness identifies the spectral set with the closure of the eigenvalues. Finally, a zero eigenfunction has zero weak gradient and is constant on connected Ω . Its infinite volume excludes a nonzero L^2 constant. This proves all assertions of Theorem 1.

Verification, priority and availability

The accompanying source and verification archive preserves the original proof and dated audits, independent geometric and spectral reconstructions, bounded primary-source priority searches, their adversarial challenge, exact finite diagnostics and file-integrity checks. Those computations do not formally verify the infinite-dimensional theorem or compute admissible apertures. An explicitly corrected source-reading error in one audit is preserved with its superseding addendum; it did not change the theorem.

The priority searches located no earlier full matching result. They do not certify historical firstness or worldwide current openness. Some older full bodies, a restricted finite-dimensional gap-transport note, unindexed work and nonpublic material remained outside the comparison. The dated question and substantive antecedents are credited above; the claim is the complete construction proved here. The research record is repository PR 765. This note and its authored supporting material are released under CC BY 4.0.

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