A counterexample to ideal closure of property-(s) elements in skew braces

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Abstract

The elements with property (s) in a skew left brace need not form an additive subgroup, even when the additive group is abelian. On $\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$ we give an explicit standard semidirect brace whose property-(s) set is $\mathbb{Z}\times\{0,1\}$. The element $(0,1)$ has both defining indices equal to two, whereas its additive double has both indices infinite. This answers the unrestricted ideal-closure question of Colazzo negatively. The construction uses known ingredients. We credit an exact overlapping October 2026 preprint and record our earlier September 2026 repository disclosure; historical priority and mutual independence remain unresolved.

1 The question and exact conventions

A skew left brace is a set B with group operations $+$ and $\circ$ satisfying

$$a \circ (b + c) = (a \circ b) - a + (a \circ c).$$

The common identity is denoted by 0; addition need not be commutative. Write $\lambda_a(b) = -a + a \circ b$ and $a * b = \lambda_a(b) - b$. For $z \in B$, put

$$H_z = \{g : z * g = 0\} \cap C_{(B,+)}(z), \qquad K_z = \{g : g * z = 0\} \cap C_{(B,\circ)}(z).$$

An element has *property (s)* when both

$$[(B,+) : H_z] < \infty, \qquad [(B,\circ) : K_z] < \infty. \tag{1}$$

These are the right and left fixed intersections in the original definition; their ambient groups are different.

Colazzo's Question 1 in Oberwolfach Report 9/2023, printed p. 543 [1], asks whether the set $\mathcal{S}(B)$ of these elements is an ideal for an arbitrary skew brace. The question imposes no finiteness, global property-(S), finite-generation or two-sided assumption. In a finite brace every element has (s). Colazzo–Ferrara–Trombetti's published Definition 3.7 uses (1); Proposition 3.10 proves ideality for two-sided braces and the following text leaves the general case open [2]. An ideal must in particular be an additive subgroup.

2 The counterexample and complete index calculation

On $C = \mathbb{Z}/4\mathbb{Z}$ set

$$c \diamond d = c + d + 2cd \pmod 4, \qquad \varepsilon(0) = \varepsilon(1) = 1, \quad \varepsilon(2) = \varepsilon(3) = -1.$$

Residues in statements involving $\{0,1\}$ are represented by $0, 1, 2, 3$.

Theorem 1. *Give $B = \mathbb{Z} \times C$ coordinatewise addition and multiplication*

$$(n, c) \circ (m, d) = (n + \varepsilon(c)m,\ c \diamond d). \tag{2}$$

Then B is a left brace with abelian additive group, and

$$\mathcal{S}(B) = \mathbb{Z} \times \{0, 1\}.$$

For $x = (0, 1)$ both indices in (1) equal 2. For $x + x = (0, 2)$ both are infinite. Thus $\mathcal{S}(B)$ is neither an additive subgroup nor an ideal. It also fails λ-invariance, although it is a normal subgroup of $(B, \circ)$.

Proof. Write $c = u + 2v$, $d = s + 2t$, with all four bits in $\{0, 1\}$. Modulo four,

$$c \diamond d = (u + s \bmod 2) + 2(v + t \bmod 2).$$

Indeed the carry from $u + s$ is canceled modulo four by $2us$. Hence $(C, \diamond)$ is the Klein four group and $\varepsilon(c) = (-1)^v$ is a character of this group. Equation (2) is its action by signs on $\mathbb{Z}$, so it is an associative group law with identity $(0, 0)$ and inverse

$$(n, c)_\circ^{-1} = (-\varepsilon(c)n, c).$$

In particular, this is a group construction over all integers.

Direct calculation gives

$$\lambda_{(n,c)}(m, d) = (\varepsilon(c)m, (-1)^c d), \tag{3}$$

$$(n, c) * (m, d) = ((\varepsilon(c) - 1)m, 2cd). \tag{4}$$

The second coordinates are modulo four. Each map in (3) is an additive automorphism. Together with $a \circ b = a + \lambda_a(b)$ this proves the left brace identity.

For $z = (n, c)$ and $g = (m, d)$, addition is abelian, so

$$H_z = \{(m, d) : (\varepsilon(c) - 1)m = 0, \quad 2cd = 0\}, \tag{5}$$

$$K_z = \{(m, d) : (\varepsilon(c) - 1)m = 0, \quad (\varepsilon(d) - 1)n = 0, \quad 2cd = 0\}. \tag{6}$$

To verify (6), $g * z = 0$ imposes $(\varepsilon(d) - 1)n = 0$ and $2cd = 0$. The remaining commutation equation is $(\varepsilon(c) - 1)m = (\varepsilon(d) - 1)n$, which then reduces to the first equation in (6).

For $c = 0, 1$, these groups have the form $\mathbb{Z} \times L$. Their exact indices are

z	$[(B, +) : H_z]$	$[(B, \circ) : K_z]$
$(0, 0)$	1	1
$(n, 0),\ n \neq 0$	1	2
$(0, 1)$	2	2
$(n, 1),\ n \neq 0$	2	4.

For the second index the respective subsets L are $C, \{0, 1\}, \{0, 2\}, \{0\}$, all subgroups of $(C, \diamond)$. The projection $(B, \circ) \to (C, \diamond)$ is surjective, so these indices are $4/|L|$. For the first index, (5) gives B when $c = 0$ and $\mathbb{Z} \times \{0, 2\}$ when $c = 1$.

If $c = 2$ or 3, both (5) and (6) force $m = 0$. The additive cosets represented by $(j, 0)$, $j \in \mathbb{Z}$, are pairwise distinct, since their differences have nonzero first coordinate. The multiplicative cosets are also pairwise distinct: $(i, 0)_\circ^{-1} \circ (j, 0) = (j - i, 0)$ belongs to K_z only when $i = j$. This proves both infinite indices and the asserted complete set, without extrapolation from finite computations.

Finally $x + x = (0, 2)$, while $\lambda_x(x) = (0, 3) \notin \mathcal{S}(B)$. The character $(n, c) \mapsto \varepsilon(c)$ on $(B, \circ)$ has kernel $\mathcal{S}(B)$, proving multiplicative normality. □

The circle group is $D_\infty \times C_2$: the high bit acts by inversion on $\mathbb{Z}$, while the low bit acts trivially. The brace is not two-sided. For $u = v = (0, 1)$ and $w = (1, 0)$,

$$(u + v) \circ w = (-1, 2), \qquad (u \circ w) - w + (v \circ w) = (1, 2).$$

Thus the counterexample respects the two-sided restriction in the known positive result.

3 Known ingredients, overlap and priority limits

The finite brace $(C, +, \diamond)$ is old: Bachiller's Proposition 2.4(iii) includes it [3], and CFT Example 3.8 recalls it [2]. The semidirect construction is standard, for example Smoktunowicz–Vendramin's published Corollary 3.37 [4]. Dietzel's Theorem 5(iv) explicitly uses the same high-bit inversion character in a finite cyclic-module family [5]. Replacing that trivial finite module by the trivial brace $\mathbb{Z}$ and making the index calculation above is an elementary specialization. We claim no new construction method.

The example is also an immediate application of the earlier finite-conjugacy machinery once the character is chosen. Cascella–Properzi–Van Antwerpen's Proposition 5.17 identifies $\mathcal{S}(B)$ with the multiplicative FC-centre when every element has finite additive conjugacy and a finite λ-orbit [6]. Those hypotheses hold here: addition is abelian and the λ image has four elements. Conjugation is explicitly

$$(m, d) \circ (n, c) \circ (m, d)_{\circ}^{-1} = ((1 - \varepsilon(c))m + \varepsilon(d)n, c),$$

which gives the same set. The direct proof of Theorem 1 does not depend on that characterization.

Di Matteo–Ferrara also give a pre-September characterization and a finite-factor construction [7]. An infinite-module variation of their Section 4, and another of Mondal–Shumyatsky–Trombetti–Yadav's finite Theorem 4.7 [8], give alternative negative examples by short calculations. The supporting priority audit records those derivations. They are deductions made in our audit, not claims that those papers explicitly published the unrestricted negative answer. This ancestry limits any broad originality claim.

Alper Ferudun's unrefereed October 11, 2026 note, Theorem 1.2 [9], gives exactly the same construction and property-(s) set as Theorem 1. We credit this exact overlap. GitHub records our PR 156 as created on September 30, 2026 at 12:24:37 UTC [10]; its original proof is unchanged. The first proof commit is dated September 30 at 12:16:29 UTC, an author-supplied commit timestamp distinct from the provider's PR date. Ferudun's first Zenodo record was created on October 11 at 05:15:40.640397 UTC. This note's formal preprint date is later and is not backdated. The repository was public according to the author; current public visibility was checked, while historical visibility was not independently archived.

We describe these as *contemporaneous separate discoveries*, meaning separate disclosures in the same period. The dates establish an ordering of the recorded disclosures, not a literally simultaneous release or either author's awareness of the other's work. Mutual independence has not been established.

Our bounded primary-source audit found no earlier explicit negative answer in the full bodies read. It does not certify historical novelty, first priority or exclusive resolution. Several complete publisher bodies were unavailable: JKVV's 2021 *Radical and weight* article, the 2026 CPV and Mondal–Yadav articles, and the 2026 word/conjugacy-problem article (DOI 10.1080/10586458.2025.2598349). Available full preprints or the author-labeled CPV postprint were read where available. The precise versions, additional access gaps and bounded search limits are preserved in the supporting audit. These gaps can affect priority assessment; they do not affect the self-contained counterexample.

Verification and disclosure

The support archive contains the standalone source, an exact standard-library checker, recorded ordinary and optimized runs, negative controls, source locators and the priority/provenance audit. The checker uses explicit raised errors, so optimization does not disable its checks. Its finite and symbolic controls supplement the written proof; the infinite-coset argument is the

certificate of infinite index. Every finite quotient has property (s) globally and cannot by itself establish this failure.

AI tools were used extensively in solving, drafting, literature comparison, reproduction and adversarial verification. This preprint is unrefereed and has not undergone conventional human peer review. The author is responsible for its contents.

References

[1] I. Colazzo, Derived-indecomposable solutions and skew braces whose elements have a finite number of conjugates, contribution to *Oberwolfach Report* 9/2023, pp. 543–544, Question 1. doi:10.4171/OWR/2023/9.

[2] I. Colazzo, M. Ferrara and M. Trombetti, On derived-indecomposable solutions of the Yang–Baxter equation, *Publ. Mat.* 69 (2025), 171–193. doi:10.5565/PUBLMAT6912508.

[3] D. Bachiller, Classification of braces of order p^3, *J. Pure Appl. Algebra* 219 (2015), 3568–3603. doi:10.1016/j.jpaa.2014.12.013; arXiv:1407.5224v1.

[4] A. Smoktunowicz and L. Vendramin, On skew braces (with an appendix by N. Byott and L. Vendramin), *J. Comb. Algebra* 2 (2018), 47–86. doi:10.4171/JCA/2-1-3.

[5] C. Dietzel, Braces of order p^2q, *J. Algebra Appl.* 20 (2021), 2150140. doi:10.1142/S0219498821501401; arXiv:1801.06911v2, Theorem 5(iv).

[6] R. Cascella, S. Properzi and A. Van Antwerpen, Finiteness conditions on skew braces and solutions of the Yang–Baxter equation, *Mediterr. J. Math.* 23 (2026), article 187. doi:10.1007/s00009-026-03182-4. Full author-labeled postprint: arXiv:2603.06177v2, submitted September 14, 2026.

[7] M. Di Matteo and M. Ferrara, Finite-index problems in skew braces, arXiv:2607.20222v1, July 22, 2026.

[8] S. Mondal, P. Shumyatsky, M. Trombetti and M. K. Yadav, Relative commuting probability and BFC-type results for finite skew braces, arXiv:2609.16269v1, September 14, 2026.

[9] A. Ferudun, The Elements with Property (s) of a Skew Brace Need Not Form an Ideal: On a Question of Colazzo, Ferrara and Trombetti, unrefereed preprint, version 1, October 11, 2026. doi:10.5281/zenodo.23292254.

[10] A. Kriebel, Original counterexample draft, PR 156, github.com/AlecKriebel/Math/pull/156, September 30, 2026. Initial proof commit `c1677fe1ed901b8d9c5efe4cbd3d3fec7d0ba227`.