# A projective-line quotient of the Rogalski–Sierra algebra

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**Abstract**

For the six-relation presentation $R(\alpha, \beta)$ in Rogalski and Sierra's 2010 report and 2012 article, the linear ideal $(x_3 - x_1, x_4 - x_2)$ has a projective-line twisted homogeneous coordinate ring as its exact quotient. It is a nonzero homogeneous completely prime ideal strictly below the augmentation ideal. This gives negative answers to both printed assertions: uniqueness of the nontrivial graded prime and absence of nontrivial maps to twisted homogeneous coordinate rings. The calculation works over every field whenever $\beta^2 \neq 1$, hence for all the prescribed algebraically independent parameters. The classical twist machinery is credited; historical priority of this explicit quotient remains unresolved.

## 1 The displayed problem

Let $k$ be a field, let $\alpha, \beta \in k$, and give the generators degree one in

$$R = R(\alpha, \beta) = k\langle x_1, x_2, x_3, x_4\rangle/(f_1, \ldots, f_6),$$

where

$$\begin{aligned}
f_1 &= x_1(\alpha x_1 - x_3) + x_3(x_1 - \alpha x_3),\\
f_2 &= x_1(\alpha x_2 - x_4) + x_3(x_2 - \alpha x_4),\\
f_3 &= x_2(\alpha x_1 - x_3) + x_4(x_1 - \alpha x_3),\\
f_4 &= x_2(\alpha x_2 - x_4) + x_4(x_2 - \alpha x_4),\\
f_5 &= x_1(\beta x_1 - x_2) + x_4(x_1 - \beta x_2),\\
f_6 &= x_1(\beta x_3 - x_4) + x_4(x_3 - \beta x_4).
\end{aligned}$$

In particular, the left factor in the second summand of both $f_5$ and $f_6$ is $x_4$.

The official report [1, pp. 1361–1362] gives these relations and asks whether, for algebraically independent $\alpha, \beta$ over the prime subfield of an uncountable algebraically closed $k$, the augmentation ideal $R_+ = \bigoplus_{n>0} R_n$ is the only nontrivial graded prime ideal, and whether $R$ has no nontrivial homomorphism to any twisted homogeneous coordinate ring. Both assertions recur in the published article [2, p. 1198], whose Theorem 3.5(2) on p. 1202 gives this presentation with parameters $c, d$ in place of $\alpha, \beta$. The printed assertions impose no injectivity, birationality or lower bound on the dimension of the image. We exhibit a unital graded surjection with infinite-dimensional image of Gelfand–Kirillov dimension two.

## 2 The quotient

We use positive pullbacks in the convention

$$B(X,L,\sigma)_n = H^0(X,L_n), \qquad L_n = L\otimes\sigma^*L\otimes\cdots\otimes(\sigma^{n-1})^*L, \qquad f\star g = f\otimes(\sigma^m)^*g$$

for $f$ of degree $m$, with $L_0 = \mathcal{O}_X$. This is the classical twisted homogeneous coordinate ring construction [3, 4].

**Theorem 1.** *For every field $k$, every $\alpha\in k$, and every $\beta\in k$ with $\beta^2\neq 1$, put*

$$I = (x_3-x_1, x_4-x_2)\subset R, \qquad \sigma[s:t] = [s+\beta t : \beta s + t].$$

*Then*

$$R/I \;\cong\; k\langle X,Y\rangle/(\beta X^2 - XY + YX - \beta Y^2) \;\cong\; B(\mathbb{P}^1_k,\mathcal{O}(1),\sigma).$$

*The isomorphism sends $x_1, x_3$ to $s$ and $x_2, x_4$ to $t$. Moreover, $0\neq I\subsetneq R_+$, $I$ is homogeneous and completely prime, and $R/I$ has Hilbert series $(1-z)^{-2}$ and $\mathrm{GKdim}(R/I)=2$.*

*Proof.* Imposing $x_3 = x_1 = X$ and $x_4 = x_2 = Y$ sends the six relations to $0,0,0,0,Q,Q$, where

$$Q = \beta X^2 - XY + YX - \beta Y^2.$$

The universal property of free-algebra quotients therefore gives the first displayed isomorphism, with exactly this defining ideal.

Let $S = k[s,t]$ and let $\theta$ be the graded algebra automorphism

$$\theta(s) = s+\beta t, \qquad \theta(t) = \beta s + t.$$

It is invertible since its determinant is $1-\beta^2$. On the graded vector space $S$, define $f\star g = f\theta^m(g)$ for homogeneous $f\in S_m$, and extend bilinearly. The automorphism law proves associativity and the unit property. A product of two nonzero homogeneous elements is nonzero; taking highest nonzero homogeneous components proves that this algebra $B$ is a domain for arbitrary elements as well. Its degree $n$ piece has dimension $n+1$. It is generated by degree one: inductively, $S_n\theta^n(S_1) = S_{n+1}$.

The linear lift of $\sigma$ above identifies $\sigma^*$ on $H^0(\mathcal{O}(1))$ with $\theta$, and identifies $L_n$ with $\mathcal{O}(n)$. Under these compatible identifications the section multiplication is exactly $\star$. Thus $B = B(\mathbb{P}^1,\mathcal{O}(1),\sigma)$. The line bundle is $\sigma$-ample: $L_n\cong\mathcal{O}(n)$, so Serre vanishing applies to every coherent sheaf tensored with $L_n$.

Direct computation in the commutative polynomial ring $S$ gives

$$\beta\, s\star s - s\star t + t\star s - \beta\, t\star t = 0.$$

Since $s,t$ generate $B$, there is a graded surjection $A := k\langle X,Y\rangle/(Q)\twoheadrightarrow B$. To prove injectivity without any confluence assumption, use $U = X-Y$, $V = Y$. This change of generators is invertible in every characteristic, and

$$Q = \beta U^2 + (\beta-1)UV + (\beta+1)VU.$$

Because $\beta+1\neq 0$, the relation rewrites $VU$ as

$$VU = \frac{1-\beta}{1+\beta}UV - \frac{\beta}{1+\beta}U^2.$$

Every replacement strictly decreases fixed-length lexicographic order with $V > U$. Reduction therefore terminates and shows that $U^iV^{n-i}$, $0 \le i \le n$, span $A_n$. Consequently $\dim A_n \le n+1 = \dim B_n$. The surjection forces equality in each degree, hence is an isomorphism and the kernel of $R \to B$ is exactly $I$.

The relations of $R$ are quadratic, so its four degree-one generators are linearly independent. In particular $x_3 - x_1 \neq 0$. The ideal $I$ is homogeneous, two-sided and contained in $R_+$. Since $x_1$ maps to the nonzero element $s$, it is not in $I$; thus $0 \neq I \subsetneq R_+$. The domain quotient proves complete primality, and hence both ordinary and graded primality. Finally $\dim B_n = n+1$, so the Hilbert series is $(1-z)^{-2}$; the cumulative dimension is $(n+1)(n+2)/2$, proving $\operatorname{GKdim} B = 2$. □

Algebraic independence of $\alpha, \beta$ implies $\beta^2 \neq 1$. Thus the ideal and surjection refute both assertions in the stated parameter regime. The image comes from a positive-dimensional projective curve; it is neither an augmentation to $k$ nor a finite-dimensional image.

## 3 Scope and historical context

No classification of all primes is claimed. Nor does a quotient contradict the separate obstruction to sharing a graded quotient ring with a twisted homogeneous coordinate ring discussed in [2, p. 1198]. The source's surface algebra and this curve quotient have different Gelfand–Kirillov dimensions.

The theorem concerns the displayed presentation in all characteristics. The journal's geometric identification uses $c = (\vartheta-1)/(\vartheta+1)$ and $d = (\rho-1)/(\rho+1)$ [2, Theorem 3.5]. These preserve independent parameters in characteristic different from two; in characteristic two the fractions collapse to one when defined. The theorem does not identify the independent-parameter characteristic-two presentation with that geometric family or transfer the latter's noetherian or dimension results. At the excluded values $\beta = \pm 1$, the corresponding $Q$ factors as $(X+Y)(X-Y)$ or $-(X-Y)(X+Y)$; the quotient then has nonzero linear zero divisors. At $\beta = 0$ it is the commutative polynomial plane, consistent with the stronger theorem.

The two-generator algebras, degree-reduction argument and polynomial graded twists are classical. The source-specific point is the substitution $(x_1, x_2, x_3, x_4) \mapsto (X, Y, X, Y)$ and its exact projective-line quotient. Later Nevins–Sierra preprints [5, v1, p. 25; v2, pp. 28–29] repeat the contrary no-map assertion for the Rogalski–Sierra algebra. Their Theorem 1.2 requires additional point-tail moduli and geometric hypotheses, including dimension at least two; a universal factor in Theorem 4.10 need not itself be a twisted homogeneous coordinate ring. Likewise the canonical-map theorem in [4] requires strong noetherianity, which the generic source family lacks [2, p. 1236]. These theorems alone do not supply an earlier resolution. Campbell's thesis [6, Chapter 6 and §8.2] treats infinitesimal deformations at a baseline algebra and function skew fields of another deformation family; those statements are not this exact generic algebra's homomorphic curve quotient.

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## References

[1] S. J. Sierra, *Birationally commutative projective surfaces of GK-dimension 4* (joint work with D. Rogalski), in *Oberwolfach Report 22/2010*, pp. 1360–1363. doi:10.4171/OWR/2010/22.

[2] D. Rogalski and S. J. Sierra, *Some projective surfaces of GK-dimension 4*, Compositio Math. 148 (2012), 1195–1237. doi:10.1112/S0010437X12000188.

[3] M. Artin and M. Van den Bergh, *Twisted homogeneous coordinate rings*, J. Algebra 133 (1990), 249–271.

[4] D. Rogalski and J. J. Zhang, *Canonical maps to twisted rings*, Math. Z. 259 (2008), 433–455; arXiv:math/0409405v2.

[5] T. A. Nevins and S. J. Sierra, *Naïve blowups and canonical birationally commutative factors*, arXiv:1206.0760v1 (2012), v2 (arXiv revision 2014; retrieved PDF internally dated 15 November 2018).

[6] C. J. M. Campbell, *Deformation theory of a birationally commutative surface of Gelfand–Kirillov dimension 4*, Ph.D. thesis, University of Edinburgh, 2016. hdl:1842/22886.