

POST-LIE NONEXISTENCE WITH A SEMISIMPLE TARGET

ALEC KRIEBEL

ABSTRACT. Let $\mathfrak{g}$ be a finite-dimensional complex unimodular Lie algebra and $\mathfrak{n}$ a complex semisimple Lie algebra of the same dimension. We prove that if two Lie algebra homomorphisms $j_1, j_2 : \mathfrak{g} \rightarrow \mathfrak{n}$ have invertible difference, then $\mathfrak{g}$ is semisimple. The integrated left-right action has a holomorphic infinitesimal determinant. Unimodularity makes this determinant invariant, and its constancy on an open orbit extends to the whole target group. Transitivity and simple connectedness then identify the source and target manifolds; their compact-form homology rules out a nonzero source radical. Consequently a perfect nonsemisimple complex source and a semisimple target admit no post-Lie algebra structure.

1. INTRODUCTION

All Lie algebras in this note are finite-dimensional over $\mathbb{C}$. For two brackets on a common vector space, a post-Lie product relates the source Lie algebra $\mathfrak{g}$ to the target Lie algebra $\mathfrak{n}$. Burde, reporting joint work with Dekimpe, formulated the nonexistence conjecture for a perfect nonsemisimple source and an arbitrary semisimple target in Oberwolfach Report 46/2023, printed p. 2689 [5]. Burde–Dekimpe–Monadjem proved the target $\mathfrak{sl}_3(\mathbb{C})$ case and further special cases [4, Theorem 3.10 and Propositions 3.11–3.12]. An earlier result of Burde–Dekimpe excludes a *solvable unimodular* source with a nonzero semisimple target [2, Theorem 3.3].

We prove the conjectured nonexistence statement as a consequence of a stronger obstruction for unimodular sources. The passage from post-Lie products to transverse pairs of homomorphisms is established in the literature [3, Proposition 2.16], [4, Proposition 2.6]; we include the reduction with explicit bracket conventions.

The determinant method has a classical affine-space predecessor. Baues associates a determinant relative invariant to an étale affine representation and proves that triviality of its character forces global transitivity [1, §5.4.1, equation (5.3), and Theorem 5.19]; see also his Corollary 5.20 for the unimodular, volume-preserving case. We use that method on a complex group: the determinant is globally holomorphic, and the identity theorem replaces polynomial continuation on affine space. The subsequent obstruction uses classical integration, Levi decomposition, compact real forms and ordinary singular homology. In particular, we do not invoke an affine-space completeness theorem for the group target.

The literature comparison underlying this note is bounded: no prior full theorem under the matching hypotheses was located in the primary passages inspected. This is not a claim of firstness, exhaustive coverage, or current openness. In particular, the original body of Onishchik’s 1969 work on reductive-group decompositions was inaccessible in that comparison; this is an access limitation, not an identified competing theorem.

2. A UNIMODULAR TRANSVERSE-PAIR OBSTRUCTION

Recall that $\mathfrak{g}$ is *unimodular* if $\mathrm{tr}(\mathrm{ad}_{\mathfrak{g}} x) = 0$ for every $x \in \mathfrak{g}$.

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Theorem 1. *Let $\mathfrak{g}$ and $\mathfrak{n}$ be complex Lie algebras of the same finite dimension, with $\mathfrak{g}$ unimodular and $\mathfrak{n}$ semisimple. Suppose $j_1, j_2 : \mathfrak{g} \rightarrow \mathfrak{n}$ are Lie algebra homomorphisms and $j_1 - j_2$ is a vector-space isomorphism. Then $\mathfrak{g}$ is semisimple.*

We first recall the solvable-group fact needed in the proof [6, Theorem 49.1], including its short induction.

Lemma 2. *The connected simply connected complex Lie group with a solvable Lie algebra $\mathfrak{r}$ is diffeomorphic to $\mathbb{C}^{\dim_{\mathbb{C}} \mathfrak{r}}$, and is therefore contractible.*

Proof. Induct on $\dim_{\mathbb{C}} \mathfrak{r}$. For $\mathfrak{r} \neq 0$, solvability gives a nonzero functional vanishing on $[\mathfrak{r}, \mathfrak{r}]$. Its kernel $\mathfrak{r}'$ is a codimension-one ideal. A complementary line $\mathbb{C}x$ is a subalgebra, so $\mathfrak{r} = \mathfrak{r}' \rtimes \mathbb{C}x$. Let R' be the simply connected integration of $\mathfrak{r}'$. The automorphisms $\exp(t \operatorname{ad} x)|_{\mathfrak{r}'}$, for $t \in \mathbb{C}$, integrate uniquely to automorphisms of R' . They give a holomorphic action of the additive group $\mathbb{C}$. Thus $R' \rtimes \mathbb{C}$ is simply connected and integrates $\mathfrak{r}$. Uniqueness of simply connected integration identifies it with the required group, whose manifold is $R' \times \mathbb{C}$. Induction proves the assertion. We use integration and uniqueness in their complex, finite-dimensional form [6, Theorems 9.12–9.13 and Corollary 9.14]. $\square$

Proof of Theorem 1. Put $d = \dim_{\mathbb{C}} \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{n}$. If $d = 0$, both algebras are zero and the conclusion holds. Assume $d > 0$. Choose connected simply connected complex Lie groups G, N integrating $\mathfrak{g}, \mathfrak{n}$. The maps j_i integrate to holomorphic homomorphisms $J_i : G \rightarrow N$ [6, Theorems 9.12–9.13 and Corollary 9.14]. They define a global holomorphic action

$$(1) \quad A_h(p) = J_1(h) p J_2(h)^{-1} \quad (h \in G, p \in N).$$

No closedness or injectivity of the images $J_i(G)$ is required.

Under left trivialization of TN , the differential at the identity of $h \mapsto A_h(p)$ is

$$(2) \quad B_p : \mathfrak{g} \longrightarrow \mathfrak{n}, \quad B_p x = \operatorname{Ad}_N(p^{-1}) j_1 x - j_2 x.$$

Fix bases of $\mathfrak{g}, \mathfrak{n}$ and define the globally holomorphic function $f(p) = \det B_p$. Since $B_e = j_1 - j_2$ is invertible, the holomorphic inverse function theorem makes the orbit of e open.

For $q = A_h(p)$, equivariance of the integrated homomorphisms gives

$$(3) \quad B_q = \operatorname{Ad}_N(J_2(h)) B_p \operatorname{Ad}_G(h^{-1}).$$

Indeed, the first term on the right is

$$\operatorname{Ad}_N(J_2(h)) \operatorname{Ad}_N(p^{-1}) \operatorname{Ad}_N(J_1(h)^{-1}) j_1,$$

and the second is $-\operatorname{Ad}_N(J_2(h)) j_2 \operatorname{Ad}_G(h^{-1}) = -j_2$. These are the two terms of B_q , since $q^{-1} = J_2(h) p^{-1} J_1(h)^{-1}$.

For a connected complex Lie group L , the character $\det \operatorname{Ad}_L : L \rightarrow \mathbb{C}^\times$ has differential $x \mapsto \operatorname{tr}(\operatorname{ad}_{\operatorname{Lie}(L)} x)$. If this differential is zero, the character is locally constant, hence identically one. This applies to G by unimodularity, and to N because a semisimple Lie algebra is perfect and the trace of a commutator is zero. Taking determinants in (3) yields

$$f(A_h(p)) = f(p).$$

Consequently $f = f(e) \neq 0$ on the open identity orbit. The identity theorem on the connected complex manifold N gives $f \equiv f(e)$ on all of N . Every B_p is therefore invertible, so every orbit is open. Connectedness of N implies that there is only one orbit: the action (1) is transitive.

Let $H = \{h \in G : A_h(e) = e\}$. It is a closed subgroup with Lie algebra $\ker B_e = 0$, hence is discrete. The orbit map identifies G/H with N as a manifold [6, Proposition 4.12 and Corollary 4.16]. For completeness, $G \rightarrow G/H$ is a covering: choose an identity neighborhood U with $U^{-1}U \cap H = \{e\}$; the sets Uh , $h \in H$, are disjoint sheets over the image of U , and left translations

give such neighborhoods everywhere. Because G is connected and N is simply connected, this covering has one sheet [7, Proposition 1.32]. Thus $H = \{e\}$, and

$$(4) \quad G \text{ is diffeomorphic to } N.$$

The orbit map need not be a group homomorphism; no group or Lie algebra isomorphism is inferred from (4).

Take a Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{r}$ is the solvable radical and $\mathfrak{s}$ is semisimple. Let R, S be their simply connected integrations. The Levi action integrates on R ; the simply connected group $R \rtimes S$ has Lie algebra $\mathfrak{g}$. By uniqueness of integration, its manifold $R \times S$ is the manifold of G [6, Theorem 48.9 and Corollary 9.14]. This construction imposes no closed-image assumption on a previously chosen Levi subgroup. Lemma 2 makes the R factor contractible.

Polar decomposition retracts a simply connected complex semisimple group onto a compact real form [6, Corollary 43.5, Theorem 43.7, Corollary 43.8 and the following covering discussion]. Let these compact groups be K_N for N and K_S for S . Their real dimensions are respectively d and $s = \dim_{\mathbb{C}} \mathfrak{s}$, rather than $2d$ and $2s$. The compact connected Lie group K_N is orientable: left translation of a basis at its identity gives a global frame. Its fundamental class therefore gives, in ordinary singular homology [7, Theorem 3.26(a)],

$$H_d(N; \mathbb{Z}) \cong H_d(K_N; \mathbb{Z}) \cong \mathbb{Z}.$$

If $\mathfrak{r} \neq 0$, then $s < d$. The product description of G , contractibility of R , and polar decomposition for S give

$$H_d(G; \mathbb{Z}) \cong H_d(K_S; \mathbb{Z}) = 0,$$

since K_S is a compact s -manifold [7, Theorem 3.26(c)]. This contradicts (4). Hence $\mathfrak{r} = 0$, proving that $\mathfrak{g}$ is semisimple. $\square$

3. THE POST-LIE CONSEQUENCE

Let $\mathfrak{g} = (V, [\cdot, \cdot])$ and $\mathfrak{n} = (V, \{\cdot, \cdot\})$. Our convention for a post-Lie product $x \cdot y$ is

$$(5) \quad x \cdot y - y \cdot x = [x, y] - \{x, y\},$$

$$(6) \quad [x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z),$$

$$(7) \quad x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}.$$

Thus $L(x)y = x \cdot y$ defines a homomorphism $L : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{n})$.

Corollary 3. *If a finite-dimensional complex pair $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie product, $\mathfrak{g}$ is unimodular and $\mathfrak{n}$ is semisimple, then $\mathfrak{g}$ is semisimple. In particular, a perfect nonsemisimple complex source and a semisimple target admit no post-Lie product.*

Proof. For semisimple $\mathfrak{n}$, every derivation is inner and its center is zero. To recall the innerness argument, let κ be its nondegenerate Killing form and $D \in \text{Der}(\mathfrak{n})$. Choose $a \in \mathfrak{n}$ such that $\kappa(a, u) = \text{tr}(D \text{ad}_{\mathfrak{n}} u)$. Using $[D, \text{ad}_{\mathfrak{n}} u] = \text{ad}_{\mathfrak{n}}(Du)$, cyclicity of trace and invariance of κ , we obtain

$$\kappa(Du, v) = \text{tr}(D[\text{ad}_{\mathfrak{n}} u, \text{ad}_{\mathfrak{n}} v]) = \kappa(a, \{u, v\}) = \kappa(\{a, u\}, v).$$

Hence $D = \text{ad}_{\mathfrak{n}} a$. The map $\text{ad}_{\mathfrak{n}} : \mathfrak{n} \rightarrow \text{Der}(\mathfrak{n})$ is therefore a linear isomorphism. In particular there is a unique linear map

$$R = \text{ad}_{\mathfrak{n}}^{-1} L : V \longrightarrow V, \quad x \cdot y = \{Rx, y\}.$$

Equations (6)–(7) and injectivity of $\text{ad}_{\mathfrak{n}}$ give

$$R[x, y] = \{Rx, Ry\}.$$

Equation (5) gives the signs explicitly:

$$[x, y] = \{Rx, y\} + \{x, Ry\} + \{x, y\}.$$

Expanding $\{(R + I)x, (R + I)y\}$ now shows that

$$j_1 = R + I, \quad j_2 = R$$

are homomorphisms $\mathfrak{g} \rightarrow \mathfrak{n}$, with difference I . Theorem 1 applies. Finally the functional $x \mapsto \text{tr}(\text{ad}_{\mathfrak{g}} x)$ vanishes on $[\mathfrak{g}, \mathfrak{g}]$, since $\text{ad}_{\mathfrak{g}}[x, y] = [\text{ad}_{\mathfrak{g}} x, \text{ad}_{\mathfrak{g}} y]$. A perfect source is consequently unimodular. $\square$

Remark 4. The zero-dimensional algebra is semisimple by convention, so it gives no exception to the nonsemisimple-source assertion. Unimodularity is essential to Theorem 1. For example, let $\mathfrak{g}$ have basis H, E, Z , with $[H, E] = 2E$ and Z central. In $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$, use the standard basis H, E, F . The maps

$$j_1(H) = H, \quad j_1(E) = E, \quad j_1(Z) = 0, \quad j_2(H) = j_2(E) = 0, \quad j_2(Z) = F$$

are homomorphisms with invertible difference, but $\text{tr}(\text{ad}_{\mathfrak{g}} H) = 2$. The proof makes no claim for a general real form, positive characteristic, infinite dimension, or a merely reductive target. It assumes neither algebraicity nor completeness of a locally defined action, and no bijectivity of a solvable exponential map.

VERIFICATION AND AUTHORSHIP DISCLOSURE

The accompanying archival checker reports 2,896 exact finite diagnostic conditions in each run. Those calculations are auxiliary; the proof above does not infer integration, transitivity, covering theory or homology from finite testing. AI systems were used extensively in solving, checking and writing this work. Separate AI audits examined the algebraic, global-action and topological arguments. This is an unrefereed preprint: no completed human peer review or formal machine verification is claimed.

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INDEPENDENT RESEARCHER

URL: <https://orcid.org/0009-0001-9320-500X>