

Positive tangent-bundle Seshadri constants characterize projective space

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Abstract

Let X be a smooth integral projective variety over an algebraically closed field of arbitrary characteristic. We prove that positivity of the relative Seshadri constant of T_X at one closed point implies $X \simeq \mathbf{P}^{\dim X}$, resolving the conjecture of Fulger and Murayama. An absolute minimum of ample degree among rational curves through the specified point gives a proper pointed family without boundary. Its very-free members yield surjective evaluation. A contracted marking section and line-bundle descent force Picard number one and ampleness of $-K_X$, reducing the problem to the established Fano case. No general-point or characteristic-zero assumption is used. This is an unrefereed preprint developed and checked extensively with AI tools.

1 The statement and its context

For a vector bundle E on a projective variety X , let $\rho : \mathbf{P}(E) = \mathrm{Proj}_X \mathrm{Sym} E \rightarrow X$ use the quotient convention, and write $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$. The relative Seshadri constant of Fulger–Murayama is

$$\epsilon(E; x) = \inf_{\Gamma} \frac{\xi \cdot \Gamma}{\mathrm{mult}_x(\rho_* \Gamma)}, \quad (1)$$

where Γ ranges over integral curves in $\mathbf{P}(E)$ whose images contain x and which are not contracted by ρ . If $\Gamma \rightarrow C$ has total generic degree q , then $\rho_* \Gamma = q[C]$; the degree includes its inseparable factor. Global nefness of E is not required.

Theorem 1. *Let X be a smooth integral projective variety of dimension $n \geq 1$ over an algebraically closed field k of arbitrary characteristic. If $\epsilon(T_X; x) > 0$ at a closed point $x \in X$, then $X \simeq_k \mathbf{P}^n$.*

This is [1, Conjecture 4.9], also stated in Fulger’s contribution to the 2019 Oberwolfach workshop [2, p. 3289]. Fulger–Murayama prove the Fano case in every characteristic, the surface case, and the general-point case in characteristic zero. Chang proves the smooth projective toric case [3, Theorem 0.2]. The conclusion here retains the original arbitrary point and removes the additional Fano hypothesis.

Proper pointed rational-curve families and their numerical-generation properties are standard tools; see, for example, [4, Proposition 2.20(b)]. We give the numerical argument explicitly using a genuine contracted section. The proof combines these tools with a minimum taken over *all* rational images through the given point and a characteristic-free construction of the actual marked family. The final characterization is imported from the established Fano clause [1, Proposition 4.8(1)], which uses Mori’s theorem as presented in [6, Theorem V.3.2]. The conjecture itself is never an input.

2 A complete family through the specified point

Lemma 2. *Every normalization $f : \mathbf{P}^1 \rightarrow C \hookrightarrow X$ of an integral rational curve C through x has f^*T_X ample.*

Proof. Write $f^*T_X = \bigoplus_{i=1}^n \mathcal{O}_{\mathbf{P}^1}(a_i)$, and put $m = \text{mult}_x C$. A quotient onto $\mathcal{O}(a_i)$ defines a lift $s_i : \mathbf{P}^1 \rightarrow \mathbf{P}(T_X)$ of f . Its image Γ_i is birationally parametrized by s_i : its function field lies between the equal fields $k(C)$ and $k(\mathbf{P}^1)$. Hence $\xi \cdot \Gamma_i = a_i$ and $\rho_* \Gamma_i = [C]$. Equation (1) gives $a_i/m \geq \epsilon(T_X; x) > 0$. Thus each integer $a_i \geq 1$. This argument works in every characteristic, including singular rational images, without replacing Frobenius-normalized slopes on general curves by ordinary slopes. $\square$

The proof of [1, Corollary 4.6] produces an integral rational curve through the *same* point x from the hypothesis of Theorem 1. Fix a very ample line bundle A on X , and set

$$d = \min\{A \cdot C : C \subset X \text{ is integral rational and } x \in C\}. \quad (2)$$

This is an attained positive integer.

Lemma 3. *Every geometric genus-zero stable map of A -degree d with one marking evaluating to x has smooth domain $\mathbf{P}^1$ and is birational onto its image.*

Proof. First the minimum in (2) survives any algebraically closed extension K/k . A rational image through x_K of degree $c < d$ would give a degree- c normalization map $\mathbf{P}_K^1 \rightarrow X_K$, reparametrized so that $0 \mapsto x_K$. The degree- c pointed Hom scheme is of finite type over k . Its nonemptiness would give a k -point, since k is algebraically closed. The image of that map is a rational curve through x of degree at most c , contradicting (2). Rationality of the image follows from Lüroth's theorem, also in positive characteristic.

Consider a geometric map $(g : D \rightarrow X, p)$ as in the statement. Its dual graph is a tree of projective lines. Starting at the marked component, follow components contracted to x until reaching a nonconstant component D_v . Its image C_v is a rational curve through x . If e_w is the full generic degree of each nonconstant component onto its image, then

$$d = \sum_{w \text{ nonconstant}} e_w(A \cdot C_w) \geq e_v(A \cdot C_v) \geq A \cdot C_v \geq d.$$

All summands are positive integers. There is therefore just one nonconstant component, and it maps birationally to a degree- d image. Any connected contracted subtree with $t > 0$ vertices attaches to it along one edge and has at most one marking. Its total number of special-point incidences is at most $2(t-1) + 1 + 1 = 2t$, whereas stability requires at least $3t$. Such a subtree cannot exist. This includes a marking initially on a contracted component and excludes inseparable multiple covers by their full degree. $\square$

Lemma 4. *There are an integral projective scheme H , a smooth projective-line bundle $\pi : U \rightarrow H$, a section $\sigma : H \rightarrow U$, and a surjective morphism $e : U \rightarrow X$ such that $e\sigma = x$. Every fiber map is birational onto a degree- d rational curve through x .*

Proof. Abramovich–Oort [5, Theorem 2.8 and §2.5] give a proper finite-presentation Artin stack of one-marked genus-zero maps to X of A -degree d , with a projective coarse scheme Q . Let $\mathcal{M}_x$ be the actual evaluation fiber over x . It is proper and nonempty. Lemma 3 describes all its geometric objects.

Their stabilizer group schemes are trivial. A geometric automorphism preserving a birational map is the identity on the open set where the map is an isomorphism onto its image. Infinitesimal automorphisms are vector fields on $\mathbf{P}^1$ vanishing at the mark and killed by dg .

The normalization map has generically injective differential, so such a vector field is zero. A finite group scheme with one geometric point and zero tangent space is the identity: its local maximal ideal satisfies $\mathfrak{m}/\mathfrak{m}^2 = 0$, and Nakayama applies. This also makes inertia trivial over nonreduced bases. On an atlas its finite algebra has an augmentation split by the identity section as $\mathcal{O} \oplus J$. The finite module J has zero geometric fibers and therefore vanishes by Nakayama.

Trivial inertia makes $\mathcal{M}_x$ an algebraic space [9, Tag 04SZ]. Its natural morphism to Q is proper, because $\mathcal{M}_x$ is proper over k and Q is separated. It is quasi-finite: coarse geometric points identify isomorphism classes, so each geometric fiber contains at most one point. Tags 03XX and 02LS of [9] make $\mathcal{M}_x$ a scheme finite over Q , hence projective. No commutation of coarse moduli with a closed evaluation fiber in wild characteristic is assumed.

The stack's universal curve restricts to an actual smooth proper $\mathbf{P}^1$ -family on $\mathcal{M}_x$, with marking section. The marking is a relative effective Cartier divisor. Its line bundle has fiber degree one; cohomology and base change give a rank-two pushforward and a fiberwise surjective evaluation map. This identifies the family with the quotient projective bundle of that pushforward, also over nonreduced bases. One can check the identification by noting that the induced morphism is proper and fiberwise an isomorphism, hence finite; Nakayama makes its algebra map surjective, and flatness over the base makes its defining ideal have zero fibers, so that ideal vanishes.

Choose a degree- d normalization $f : \mathbf{P}^1 \rightarrow X$ with $f(0) = x$. Let R_d be the degree- d Hom scheme with $0 \mapsto x$. Its morphism to $\mathcal{M}_x$ is the torsor of pointed parametrizations under the smooth group $\text{Aut}(\mathbf{P}^1, 0)$. Indeed the projective-line bundle with section just constructed is locally a pointed trivial bundle. By Lemma 2,

$$H^1(\mathbf{P}^1, f^*T_X(-0)) = 0, \quad H^1(\mathbf{P}^1, f^*T_X(-0 - \infty)) = 0.$$

Fixed-source deformation theory makes R_d smooth at f and evaluation at ∞ smooth there. For the latter assertion the obstruction is the second displayed group: local lifts into smooth X respecting the two prescribed values differ by a Čech cocycle in $f^*T_X(-0 - \infty)$, whose vanishing obstruction allows the lifts to glue. This is infinitesimal smoothness, not generic smoothness in characteristic zero; see also [7, Proposition 4.7].

Smooth descent along the parametrization torsor makes $\mathcal{M}_x$ smooth at $[f, 0]$. Let H be its unique reduced irreducible component there, and restrict the universal curve to $U \rightarrow H$. A neighborhood of f in R_d maps into H . Its smooth evaluation at ∞ factors through $e : U \rightarrow X$, so $e(U)$ contains a nonempty open subset of X . It is closed because U is proper; irreducibility of X gives $e(U) = X$. The total space U is integral: smoothness over reduced H gives reducedness, and flatness with geometrically integral fibers over integral H gives irreducibility. It is projective because it is a projective-line bundle over projective H . All asserted fiber degrees and birationality follow from Lemma 3. $\square$

3 The contracted section and the Fano reduction

We spell out the standard numerical-generation argument for the family above. In particular it tests curves avoiding x , as well as curves of positive genus.

Lemma 5. *For every Cartier divisor D on X , put $b = D \cdot C$, where C is the image of a fiber of $U \rightarrow H$. Then b is independent of the fiber and*

$$d[D] = b[A] \quad \text{in } N^1(X)_{\mathbf{R}}.$$

Consequently $\rho(X) = 1$.

Proof. Degrees of a line bundle on fibers of a proper flat curve family are locally constant. They are constant here since H is integral. The line bundle $L = \mathcal{O}_X(dD) \otimes A^{-b}$ thus pulls back to a bundle $M = e^*L$ of degree zero on every geometric $\mathbf{P}^1$ -fiber. Each such restriction is trivial and has $h^0 = 1$, $h^1 = 0$. Proper flat cohomology and base change give a line bundle $B = \pi_*M$ with $\pi^*B \simeq M$; the adjunction map is an isomorphism on every geometric fiber and hence everywhere. The applicable perfect-base-change result is [9, Tag 0B91]; direct fiberwise descent is also given by Tags 0EX7 and 0GKA. Pulling back by σ gives

$$B \simeq \sigma^*e^*L \simeq L|_x \otimes_k \mathcal{O}_H \simeq \mathcal{O}_H.$$

Thus e^*L is trivial.

Let $\Gamma \subset X$ be any integral curve and let η be its generic point. Surjectivity of e makes U_η nonempty. A closed point of this finite-type $k(\Gamma)$ -scheme has residue field K finite over $k(\Gamma)$, of total degree $q > 0$. Its reduced closure in U is a proper integral curve Γ' dominating Γ with generic degree q . The degree projection formula gives

$$0 = \deg(e^*L|_{\Gamma'}) = q \deg(L|_\Gamma).$$

These are integer degrees, so the conclusion remains valid even if the characteristic divides q . Hence L is numerically trivial on X , proving the asserted relation. The ample class $[A]$ is nonzero since $A \cdot C = d > 0$, so the Picard number is exactly one. $\square$

Proof of Theorem 1. Use Lemma 5 with $D = -K_X$. By Lemma 2, its degree on the original rational curve is $b = \deg f^*T_X = \sum_i a_i > 0$. Thus $-K_X \equiv cA$ with $c = b/d > 0$. For every integral subvariety $Z \subset X$ of dimension $r \geq 1$, numerical equivalence gives

$$(-K_X)^r \cdot Z = c^r A^r \cdot Z > 0.$$

For completeness, subtracting the two powers expresses their difference as $(-K_X - cA)$ intersected with $\sum_{j=0}^{r-1} (-K_X)^{r-1-j} (cA)^j \cap [Z]$, a real one-cycle class; the difference therefore has degree zero. Since X is smooth, $-K_X$ is Cartier. The Nakai–Moishezon criterion, valid in every characteristic (see [8, Theorem 1.3]), makes $-K_X$ ample. The established Fano clause [1, Proposition 4.8(1)] now gives $X \simeq_k \mathbf{P}^n$ at the original point and over the original field. $\square$

When $n = 1$, one can instead use $\epsilon(T_X; x) = \deg T_X = 2 - 2g$, so positivity forces $g = 0$. Conversely projective space has positive tangent Seshadri constant: it is 2 for $\mathbf{P}^1$ and 1 for $\mathbf{P}^n$, $n \geq 2$ [1, Example 4.2]. Smoothness is essential; the singular toric examples in [3, Example 0.4] do not satisfy this theorem's hypotheses.

Verification, priority, and AI-use disclosure

The supporting package contains the source, the original frozen candidate, reproducible integrity and finite controls, independent mathematical audit reports, and a dated priority-search ledger. The finite controls supplement the written proof; they do not formally verify the geometric theorem. A bounded primary-source priority audit on 12 October 2026 identified no earlier full resolution or explicitly matching immediate consequence. It does not certify first historical priority. The standard numerical-generation principle and the previously established special cases are credited above. The full Kollár monograph and full CMSB text were not independently obtained in this audit; the exact published Fulger–Murayama propositions used here were directly inspected. An in-preparation reference on moving Seshadri constants had no inspected public contents.

AI tools were used extensively in developing the proof, checking references and calculations, performing internal adversarial reviews, and preparing the manuscript and verification

package. Responsibility for the claims and citations rests with the author. This preprint is unrefereed; internal AI-assisted review is not external human peer review, and no formal machine-checked proof is asserted.

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