Polynomial multipliers of differentiation-hypercyclic functions

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Abstract

We give a self-contained elementary negative answer to the polynomial-multiplier question posed by Costakis in 2008. There is a dense family of entire functions hypercyclic for differentiation such that every nonconstant complex polynomial multiple fails to be hypercyclic; in fact its derivative values at zero escape in modulus. The proof uses prescribed annular bounds on derivative-normalized Taylor coefficients and relative Baire category. This is an attributed historical-answer note, without a claim of a new first solution: Nikula's older random-function theorem already implies a stronger frequently hypercyclic version, as explained below.

Let $H(\mathbb{C})$ have the topology of uniform convergence on compact sets, write $Df = f'$, and put $\|f\|_R = \max_{|z|\le R} |f(z)|$. A function is in $\mathrm{HC}(D)$ if $\{D^n f : n \ge 0\}$ is dense in $H(\mathbb{C})$. Costakis's Question 5 [1, p. 330] asks whether pf must be in $\mathrm{HC}(D)$ whenever $f \in \mathrm{HC}(D)$ and p is a nonconstant polynomial. The operation is the pointwise product $p(z)f(z)$.

Theorem 1. *There is a dense subset $B \subset H(\mathbb{C})$, contained in $\mathrm{HC}(D)$, such that for every $f \in B$ and every nonconstant $p \in \mathbb{C}[z]$,*

$$|(pf)^{(n)}(0)| \longrightarrow \infty.$$

Consequently the polynomial multipliers preserving every member of $\mathrm{HC}(D)$ are exactly the nonzero constants.

The same f works for all p; the onset of the limit may depend on p. We first prove the theorem without using random hypercyclicity. Set

$$\ell_n = (n+1)^{-1/2}, \qquad u_n = \log(n+e), \qquad X_M = \{f \in H(\mathbb{C}) : \ell_n \le |f^{(n)}(0)| \le u_n \ (n \ge M)\}.$$

Each X_M is nonempty and closed in the complete metrizable space $H(\mathbb{C})$. Indeed the baseline series $\sum_{n\ge 0} \ell_n z^n/n!$ is entire, and each derivative evaluation is continuous. The bounds $u_n \le n+1$ give normally convergent series and uniformly small factorial tails on each fixed disk.

Lemma 1. *For every integer $M \ge 0$, $\mathrm{HC}(D) \cap X_M$ is dense in X_M.*

Proof. Use the countable target polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. For such a target q, and integers $R, s \ge 1$, the relative hitting set

$$U(q, R, s) = \bigcup_{N \ge 0} \{f \in X_M : \|D^N f - q\|_R < 1/s\}$$

is open in X_M. We show it is relatively dense.

Fix $g \in X_M$ and a neighborhood $\|f-g\|_r < \varepsilon$. Choose $L \geq M$ with

$$2\sum_{n>L} u_n \frac{r^n}{n!} < \varepsilon.$$

Write $q(z) = \sum_{k=0}^{d} \beta_k z^k/k!$ (for $q = 0$, take $d = 0$). For a large $N > L$, retain the jets of g through L, use the baseline ℓ_n afterward, and replace jets $N, \ldots, N+d$ by γ_k. Choose $\gamma_k = \beta_k$ if $|\beta_k| \geq \ell_{N+k}$ and otherwise move β_k radially to modulus ℓ_{N+k}, choosing the positive boundary point when $\beta_k = 0$. Then

$$|\gamma_k - \beta_k| \leq \ell_{N+k}, \qquad \ell_{N+k} \leq |\gamma_k| \leq \max(1, |\beta_k|).$$

Since $u_n \to \infty$, these finitely many jets satisfy their upper bounds for sufficiently large N. The resulting entire function F_N lies in X_M and is within ε of g on the r-disk: both tails after L have jets bounded by u_n.

Under D^N, all retained low jets vanish. On the target block the jet errors are at most ℓ_{N+k}, and after that block the jets are the baseline. Thus

$$\|D^N F_N - q\|_R \leq \ell_N e^R \longrightarrow 0.$$

Every relative neighborhood therefore meets $U(q, R, s)$. Baire's theorem, applied inside the complete space X_M, gives a dense intersection of these hitting sets. Membership means density of the derivative orbit, since the target polynomials are dense and integer disks and reciprocal-integer errors suffice. No invariance of X_M under D is needed. □

Proof of Theorem 1. Let $f \in X_M$, $a_n = f^{(n)}(0)$, and $p(z) = \sum_{j=0}^{d} c_j z^j$ with $d \geq 1$ and $c_d \neq 0$. For $n \geq M + d$, Leibniz gives

$$(pf)^{(n)}(0) = \sum_{j=0}^{d} c_j (n)_j a_{n-j}, \qquad (n)_j = \frac{n!}{(n-j)!}.$$

The top-degree term has modulus at least

$$|c_d| \frac{(n)_d}{\sqrt{n-d+1}} \asymp_p n^{d-1/2},$$

whereas the sum of the moduli of all lower terms is at most

$$\left(\sum_{j<d} |c_j|\right) n^{d-1} \log(n+e) = o(n^{d-1/2}).$$

The reverse triangle inequality proves escape, including $d = 1$ and arbitrary complex phases. This deterministic argument holds for every p after f has been fixed; no uncountable intersection is used.

Evaluation at zero is continuous and onto $\mathbb{C}$. An escaping scalar sequence has a tail outside the unit disk; an open subdisk avoiding its finitely many earlier values remains. It is not dense, so $pf \notin \mathrm{HC}(D)$.

To see ambient density, approximate any entire target by a polynomial P. For large $M > \deg P$, the function $P + \sum_{n \geq M} \ell_n z^n/n!$ belongs to X_M and is arbitrarily close to P. Lemma 1 then gives nearby hypercyclic members of X_M. Thus $B = \bigcup_M (X_M \cap \mathrm{HC}(D))$ is dense and has the required property. Nonzero constants preserve a dense derivative orbit by scaling; the zero polynomial and every nonconstant polynomial fail to preserve the full class. □

Each X_M has empty ambient interior: it is closed and contains no polynomial, while polynomials are dense. Hence the displayed B is meagre. This is compatible with Costakis's Theorem 6 [1, p. 330], which gives a residual common family on which every nonzero polynomial multiple is hypercyclic.

Earlier implication and priority. The existence conclusion of Theorem 1 already follows from Nikula's random-function theorem [2, arXiv v1, Thm. 1, pp. 2–3], even with frequent hypercyclicity. Here frequent hypercyclicity means positive lower density of visits to every nonempty open set. For IID standard circular complex Gaussian jets X_n, that theorem gives an almost surely entire, frequently hypercyclic series $f(z) = \sum_{n\geq 0} X_n z^n/n!$. Its full-support and logarithmic-tail hypotheses hold for the density $\pi^{-1}e^{-|z|^2}$.

Indeed, $\Pr(|X| < t) \leq t^2$ and $\Pr(|X| > t) = e^{-t^2}$. For $\alpha = 3/4$, Borel–Cantelli gives, on one probability-one event,

$$(n+1)^{-\alpha} \leq |X_n| \leq \log(n+e) \quad \text{eventually.}$$

Intersect with the frequent-hypercyclicity event. For any polynomial of degree $d \geq 1$, the same Leibniz estimate has leading size $n^{d-\alpha}$ and lower-term bound $O_p(n^{d-1}\log n)$; their ratio tends to zero. The event was chosen independently of p, so it supplies all-polynomial escape simultaneously. Polynomial additions preserve both frequent hypercyclicity and all sufficiently high jets, giving a dense family from a single realization.

Even the exact annular existence in Lemma 1 is a consequence of that older theorem and elementary clipping. Radially project jets $n \geq M$ into $[\ell_n, u_n]$ in modulus. Upper violations occur only finitely often almost surely; lower corrections are at most ℓ_n. The correction jets c_n therefore tend to zero, and their entire series h satisfies

$$\|D^n h\|_R \leq e^R \sup_{k\geq n} |c_k| \longrightarrow 0.$$

Such a perturbation preserves frequent hypercyclicity by retaining visits to smaller target neighborhoods after a finite prefix. The clipped independent coordinates have full relative support on their annuli (at $n = 0$, Gaussian phase gives full support on the unit circle), and the common factorial-tail bound makes finite-coordinate cylinders a neighborhood basis in X_M. Thus frequently hypercyclic functions are dense in each X_M. Nikula is applied before clipping; the clipped laws are not IID with full-plane support.

A still earlier general Gaussian-eigenfield route is supplied by Bayart–Matheron [3, arXiv v1, Prop. 3.3]: on the unit circle with Haar measure use $E(\lambda)(z) = e^{\lambda z}$. Its integral operator sends Fourier basis vectors to $z^n/n!$ or zero; the Gaussian series converges normally almost surely, has full support, and the Fourier-decay hypothesis follows from the Riemann–Lebesgue lemma. The resulting mixing Gaussian law gives almost-sure frequent hypercyclicity. The supporting audit records the specialization and its limits.

We do not claim that these sources explicitly state the polynomial-multiplier corollary, nor determine its first explicit recording or earliest underpinning. This note provides a self-contained elementary proof of the historical negative answer with the prior implication credited; it does not claim a new first solution, new existence theorem, or a problem still open in 2026. The repository's original finer fixed-polynomial null-orbit perturbation argument is retained in the supplement, without a novelty claim.

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References

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