A symmetric ideal chain with periodic Lefschetz behavior in positive characteristic

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Abstract

For each fixed field of positive characteristic p, we construct a single compatible symmetric chain of Artinian ideals in two rows of variables, generated from width three. At every width $n \geq 2$, its quotient has both the weak and strong Lefschetz properties exactly when $p \nmid n-1$. The quadratic quotient is free over the integers, and a determinant for an arbitrary linear form gives the classification over every field extension. In particular, over the fixed infinite field $\mathbb{F}_p(t)$ both properties hold and fail at unbounded widths. This gives a negative answer to the unrestricted eventual dichotomies posed by Römer. It gives no negative answer in characteristic zero or in one row.

1 The question and the chain

Let K be a fixed field and $c \geq 1$ a fixed integer. In the setup of Römer [1, pp. 1594–1596], $R_n = K[X_{i,j} : 1 \leq i \leq c,\ 1 \leq j \leq n]$ and $\mathrm{Sym}(n)$ permutes the column index j, simultaneously in every row. A symmetric invariant chain consists of ideals $I_n \subseteq R_n$ such that each I_n is $\mathrm{Sym}(n)$-invariant and

$$\mathrm{Sym}(n+1)(I_n R_{n+1}) \subseteq I_{n+1}.$$

For Artinian homogeneous quotients, Römer asks whether each Lefschetz property eventually always holds or eventually always fails. His contribution fixes an arbitrary field and $c \geq 1$; the separate one-row monomial working-group discussion on pp. 1544–1545 is a special setting.

We use the usual definitions: a standard graded Artinian K-algebra A has WLP if some $\ell \in A_1$ makes every map $\ell : A_d \to A_{d+1}$ have rank $\min(\dim A_d, \dim A_{d+1})$. It has SLP if one ℓ has this property for every $\ell^s : A_d \to A_{d+s}$, $d \geq 0$, $s \geq 1$. These are existence definitions over K. Over an infinite field the successful forms form a nonempty Zariski-open set, when they exist, since the conditions are finitely many nonvanishing minors. Standard background on characteristic-sensitive Lefschetz maps is given in [3]. Our successful forms are already defined over the prime field.

Set $c = 2$, write the variables as $x_1, \ldots, x_n, y_1, \ldots, y_n$, and put $\Updownarrow_n = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. Define

$$I_n = \Updownarrow_n^3 + (x_i x_j : i \neq j) + (y_i y_j : 1 \leq i, j \leq n) + (x_i y_i : 1 \leq i \leq n) + (x_i y_j - x_i y_k : i, j, k \text{ pairwise distinct}). \tag{1}$$

Here $\Updownarrow_n^3$ means all cubic monomials; in particular repeated variables are included. Empty index conditions contribute no generators. All generators are homogeneous and the entire list is preserved by column permutations. Every injection of column sets carries a displayed generator to one of the same type. This proves the chain compatibility, including the first two widths. Since $\Updownarrow_n^3 \subseteq I_n$, every $A_n = R_n/I_n$ is Artinian.

For $n \geq 3$, in fact $I_n = (\mathrm{Sym}(n) I_3) R_n$. One containment follows from compatibility. For the other, every generator in (1) involves at most three columns, including every cubic. Inject those distinct supporting columns into $\{1, 2, 3\}$, without changing which indices repeat. The resulting generator belongs to I_3. The reverse injection on its support extends to a permutation of $\{1, \ldots, n\}$ by bijecting the unused columns. Thus the generator is a translate of one at width three. This uses one fixed finite list of orbits, rather than choosing ideals according to the residue of n.

Theorem 1. For every field K, the quotients in (1) satisfy the following. At $n = 1$, A_1 has SLP. At every $n \geq 2$, its Hilbert function is $(1, 2n, 2n)$, and

$$A_n \text{ has SLP} \iff A_n \text{ has WLP} \iff (n-1)\, 1_K \neq 0.$$

These assertions remain true after any field extension.

2 Integral basis and multiplication

The same generators define an ideal over $\mathbb{Z}$. In degree two first remove the monomials $x_i x_j$ with $i \neq j$, all $y_i y_j$, and all $x_i y_i$. For $n \geq 2$ the remaining symbols are x_i^2 and $x_i y_j$ with $j \neq i$. Fix i and choose one reference $j_i \neq i$. The $n-1$ symbols $x_i y_j$ have relations precisely all their pairwise differences. The reference symbol, together with $x_i y_j - x_i y_{j_i}$ for $j \neq i, j_i$, is a unimodular basis for their free abelian group. The relation subgroup is exactly the span of those differences. Its quotient is therefore free of rank one, with basis $v_i = [x_i y_{j_i}]$. When $n = 2$ there is one symbol and no differences. Different i do not interact, and the $u_i = [x_i^2]$ are unrestricted. Consequently the degree-two quotient has the integral basis

$$u_1, \ldots, u_n, v_1, \ldots, v_n.$$

There are no linear relations, and all degrees at least three vanish. Thus the whole integral graded quotient is torsion free and has graded ranks $(1, 2n, 2n)$. Base change to any field gives exactly the claimed basis and Hilbert function, without a characteristic-dependent dimension argument.

Fix $n \geq 2$ and an arbitrary linear form

$$\ell = \sum_{i=1}^{n} (a_i x_i + b_i y_i), \qquad B = \sum_{i=1}^{n} b_i.$$

The defining products give, for each j,

$$\ell x_j = a_j u_j + (B - b_j) v_j, \qquad \ell y_j = \sum_{i \neq j} a_i v_i.$$

Let $D(a)$ and $D(B-b)$ denote the corresponding diagonal matrices and let J be the n by n all-ones matrix. In the ordered source basis $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and target basis $(u_1, \ldots, u_n, v_1, \ldots, v_n)$, multiplication is

$$M_\ell = \begin{pmatrix} D(a) & 0 \\ D(B-b) & D(a)(J-I) \end{pmatrix}, \qquad \det M_\ell = (-1)^{n-1}(n-1) \prod_{i=1}^{n} a_i^2. \tag{2}$$

For completeness, expand $\det(J - I)$ multilinearly in its columns over $\mathbb{Z}$. Choosing no column from J contributes $(-1)^n$. Choosing exactly one contributes $(-1)^{n-1}$ for each of the n choices. Choosing at least two gives equal columns and determinant zero. Their sum is $(-1)^{n-1}(n-1)$. This proves (2) over every field, without division by n or an eigenbasis assumption.

3 Bad widths, good widths, and the initial width

If $(n-1)1_K = 0$, equation (2) is zero for *every* choice of coefficients. Since $(A_n)_1$ and $(A_n)_2$ both have dimension $2n$, no multiplication between them can have maximal rank. There is also a direct obstruction independent of the determinant:

$$z = \sum_{j=1}^{n} y_j \neq 0, \qquad y_i z = 0, \qquad x_i z = (n-1)v_i = 0.$$

The nonzero element z is in the degree-one socle, so every linear form annihilates it. Its nonvanishing follows from the degree-one basis, regardless of whether the characteristic divides n. This obstruction also survives every field extension. In particular, failure is not inferred merely from testing one invariant form.

If $(n-1)1_K \neq 0$, take $\ell = x_1 + \cdots + x_n$. Then $M_\ell = \operatorname{diag}(I, J-I)$ is invertible. The map $(A_n)_0 \to (A_n)_1$ is injective, and maps with zero target have maximal rank. Thus ℓ is weak Lefschetz. Furthermore,

$$\ell^2 = \sum_{i=1}^{n} u_i \neq 0,$$

since all off-diagonal $x_i x_j$ vanish and the u_i are independent. This remains valid in characteristic two; it does not rely on a nonzero cross-term coefficient. The only additional map with nonzero source and target needed for SLP is $\ell^2 : (A_n)_0 \to (A_n)_2$, which is injective. All other positive powers either were checked or have zero target. Thus ℓ is strong Lefschetz. Conversely, SLP includes the maps for $s = 1$ and implies WLP.

At $n = 1$, the quotient is $K[x,y]/(xy, y^2, (x,y)^3)$, with basis $1, x, y, x^2$ and Hilbert function $(1,2,1)$. Multiplication by x from degree zero is injective, from degree one to two is surjective, and multiplication by x^2 from degree zero to two is nonzero. All remaining positive-power maps have zero target. Hence x is strong Lefschetz. This completes the proof of Theorem 1.

Corollary 2. For each prime p, over the single fixed infinite field $K = \mathbb{F}_p(t)$, the chain (1) has both WLP and SLP at every width $n = 2 + kp$ with $k \geq 0$, and neither property at every width $n = 1 + kp$ with $k \geq 1$. Consequently neither unrestricted eventual dichotomy in Römer's question holds.

Proof. The theorem applies to all widths of this one chain over this one field. Both displayed sequences are unbounded. Thus neither eventual constant truth value is possible for either property. □

For $n \geq 2$ the full pattern has period p: only $n \equiv 1 \pmod{p}$ is bad. In characteristic two, even widths are good and odd widths at least three are bad. In characteristic zero this construction is strong Lefschetz at every width. No conclusion about all characteristic-zero chains or the one-row question follows. If desired, for a fixed $c > 2$ one can add every variable in each extra row as a linear generator; the quotient is the same. The two-row example alone disproves the formulation allowing arbitrary fixed $c \geq 1$.

4 Related work, provenance, and verification limits

Seceleanu and Şega's *General symmetric ideals* [2] studies general ideals in a single row. Its Theorem B proves WLP under the hypothesis $\operatorname{char} K = 0$ or $\operatorname{char} K > n$, together with its stated size conditions. Its chain result, Theorem 9.2 (summarized in Theorem C), assumes characteristic zero. A fixed positive characteristic cannot remain larger than unbounded n. These results therefore do not give the two-row fixed-characteristic construction above. The underlying use of ranks, socles, and characteristic-sensitive determinants is standard; no new general Lefschetz mechanism is claimed.

The original primary question, pp. 1594–1596 of [1], and the introduction, Theorem B, chain setup, and Theorem 9.2 of the 18 June 2025 arXiv version of [2] were read for this comparison. The latter paper's entire proof was not audited and none of its theorems is needed in the proof here. Bounded literature and public-record searches on 22 October 2026 authenticated no earlier publication of this exact construction. This is not an exhaustive novelty or exclusive-priority certificate, and no claim about the present open status of stronger variants is made. The project's earlier proof is the unpublished PR1337, head `aa737e6`; it is a source of this note, not an earlier published solution. The internal catalog identifier is 30004534; the original-question DOI is not a solution DOI.

The accompanying source and verification archive contains a new standard-library exact checker. It independently constructs small quadratic relation quotients by integral unit-pivot elimination and checks multiplication ranks over finite fields, including exhaustive small coefficient sets, injection compatibility, width-three generation, and deliberately false controls. These finite checks supplement the universal written proof; they do not certify priority or replace a proof for arbitrary widths and fields. Source hashes, exact read scope, licenses, expected outputs, and offline reproduction instructions are included.

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