The outer-pedal area centroid in even-period elliptic billiards

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Abstract

Reznik, Garcia and Koiller observed that the area centroid of the outer-side pedal polygon of an even-period elliptic billiard is independent of the orbit phase (invariant k307). We prove this observation on the natural domain of the signed-area centroid and give its explicit value. The proof uses antipodal pairing, support coordinates and telescoping edge moments. For simple even orbits the centroid exists for every fixed pedal point. For even star orbits it can be undefined on a fixed circle, which we determine exactly and realize in a noncircular eight-period example. The note includes a reproducible verification package; the written identities give the all-period proof.

1 The statement and its domain

Let the billiard boundary and its fixed caustic be

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad E_c : \frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1, \qquad a > b > 0, \quad 0 < \lambda < b^2.$$

Assume this pair supports a Poncelet billiard family of least even period $N \geq 4$. Primitive star trajectories are allowed. The vertices P_i are ordered along the trajectory, with indices modulo N. The outer polygon has as its side lines the tangents to E at the P_i. For a fixed point M, let Q_i be its perpendicular foot on that full tangent line. For any ordered polygon q_i, identify the plane with $\mathbb{C}$ and set

$$S(q) = \frac{1}{2}\sum_i [q_i, q_{i+1}], \qquad \mathcal{T}(q) = \sum_i [q_i, q_{i+1}](q_i + q_{i+1}), \qquad C(q) = \frac{\mathcal{T}(q)}{6S(q)}, \tag{1}$$

where $[p, q] = \mathrm{Im}(\overline{p}q)$, and C is defined only if $S \neq 0$. These are signed moments, including multiplicity for self intersections. They agree with the area-centroid convention of [1]; they are neither the arithmetic vertex mean nor an unsigned union-of-regions centroid.

Put

$$\begin{aligned} H &= (a^2+b^2)/2, & e &= (a^2-b^2)/2,\\ d &= 1-\lambda(a^{-2}+b^{-2}), & F &= Hd + 2\lambda,\\ v &= \frac{-4e\lambda d}{F^2 - e^2 d^2}, & z &= M_x + iM_y,\\ R^2 &= |z|^2, & G &= 2F + dR^2. \end{aligned}$$

Write $S(M) = S(Q(M))$ for the chosen member of the family.

Theorem 1. *Orient the billiard counterclockwise around its elliptic caustic. Then* $F > 0$, $F^2 - e^2d^2 > 0$, $S(0) > 0$, *and*

$$S(M) = S(0)\frac{2F + d|z|^2}{2F}. \tag{2}$$

Whenever $G \neq 0$, *the signed area centroid is*

$$C(Q(M)) = \frac{z}{2} - \frac{z(F + 2Hd + evd) + \overline{z}(2Fv + 3ed + vd|z|^2/2)}{3(2F + d|z|^2)}. \tag{3}$$

Thus it is independent of the orbit phase wherever defined. If $d \geq 0$, *it is defined for every* M. *If* $d < 0$, *its exact undefined locus is the phase-independent circle* $|M|^2 = -2F/d$. *Least period four is equivalent to* $d = 0$, *when the centroid is* $M/3$. *Every simple winding-one even family has* $d \geq 0$.

The theorem proves the k307 observation in Table 4 and Section 3.4 of [1], with the denominator qualification already inherent in its signed centroid definition. An unqualified assertion that a centroid exists for every M in every even star family would be false. Equation (2) does not assert that $S(0)$ is phase independent. Reversal changes both signed moments by a sign and leaves the centroid unchanged. Repeated traversals of an even least-period orbit give the same centroid; repeating an odd primitive orbit does not create the antipodal hypothesis. Hyperbolic and degenerate caustics, and the circular boundary case, are outside the theorem.

The originating observation is due to Reznik–Garcia–Koiller. Bialy–Tabachnikov [2, Theorems 4.1 and 4.3] prove the neighboring vertex-centroid invariant k306 and recall the classical even-period symmetry. Schwartz–Tabachnikov [3] study signed centroid loci of Poncelet polygons between conics. We do not claim novelty for those methods, the symmetry, or the cyclic lemma below. Two independent bounded primary-source audits completed on 13 October 2026 found no earlier proof covering the present full statement; this is not a certification of historical firstness.

2 Support coordinates and antipodal pairing

Let T be the counterclockwise Poncelet map on E with the specified tangent choice, and $A(P) = -P$ its half-turn. The porism gives $T^N = \mathrm{id}$, and A commutes with T. The finite group generated by them consists of orientation-preserving circle homeomorphisms and is cyclic. Indeed a finite order orientation-preserving circle homeomorphism fixing a point is the identity: after cutting there, a moved point has strictly monotone iterates. The group therefore acts faithfully on a finite orbit, preserving its circular order, and embeds into its cyclic permutation group. Uniqueness of the involution gives $T^{N/2} = A$. Consequently

$$P_{i+N/2} = -P_i. \tag{4}$$

Let $n_i = (\cos\psi_i, \sin\psi_i)$ be the outward unit normal and $h_i > 0$ the support number. Then

$$h_i^2 = H + e\cos 2\psi_i, \qquad n_{i+N/2} = -n_i, \qquad h_{i+N/2} = h_i.$$

The caustic strictly contains the origin. Each oriented tangent chord has its next endpoint less than a half-turn ahead on E. The increasing Gauss map commutes with the half-turn, so successive lifted normal angles satisfy $0 < \Delta_i = \psi_{i+1} - \psi_i < \pi$. In particular the outer tangent intersections are finite, and

$$S(0) = \frac{1}{2}\sum_i h_i h_{i+1}\sin\Delta_i > 0. \tag{5}$$

This uses trajectory order and applies to stars too.

For consecutive eccentric angles u, w, put $\tau = (u+w)/2$ and $q = (w-u)/2 \in (0, \pi/2)$. Their chord is $(\cos\tau/a)x + (\sin\tau/b)y = \cos q$. Tangency to E_c gives $\sin^2 q = \lambda(\cos^2\tau/a^2 + \sin^2\tau/b^2)$, equivalently

$$d = [1 + \lambda(a^{-2} - b^{-2})]\cos u \cos w + [1 - \lambda(a^{-2} - b^{-2})]\sin u \sin w.$$

Since the normal is $h_i(\cos u/a, \sin u/b)$, this yields

$$dh_ih_{i+1} = F\cos\Delta_i + ed\cos(\psi_i + \psi_{i+1}). \tag{6}$$

All denominators in the theorem are controlled by

$$F + ed = a^2 - \lambda(a^2/b^2 - 1) > b^2 > 0, \qquad F - ed = b^2 + \lambda(1 - b^2/a^2) > 0.$$

In particular $F > 0$ and $F^2 - e^2d^2 > 0$.

If $N = 4$, apply (6) to (n_0, n_1) and (n_1, n_2). Their right sides negate because $n_2 = -n_0$, whereas both left sides are dh_0h_1. Hence $2dh_0h_1 = 0$ and $d = 0$. Conversely $d = 0$ makes $\cos\Delta_i = 0$, so each normal advances $\pi/2$ and the least period is four. Its outer polygon is a rectangle. In an orthonormal side frame its pedal vertices are

$$(A, M_y),\ (M_x, B),\ (-A, M_y),\ (M_x, -B), \qquad A, B > 0.$$

Direct shoelace gives $S = 2AB$ and $\mathcal{T} = 4ABM$, proving both formulas and $C = M/3$. For the remaining proof assume $d \neq 0$ and set $D = F/d$. Then $D^2 - e^2 > 0$ and (6) reads

$$h_ih_{i+1} = D\cos\Delta_i + e\cos(\psi_i + \psi_{i+1}). \tag{7}$$

3 A doubled-normal telescope

Regard n_i as a unit complex number and put $Z_i = n_i^2$. Squaring (7) and substituting the support squares gives

$$(D^2 - e^2)\cos^2\Delta_i = H^2 - e^2 + e(H - D)(\cos 2\psi_i + \cos 2\psi_{i+1}).$$

Thus, with $u = 2(H^2 - e^2)/(D^2 - e^2) - 1$,

$$\operatorname{Re}(\overline{Z}_iZ_{i+1}) = u + v(\operatorname{Re} Z_i + \operatorname{Re} Z_{i+1}), \qquad v = \frac{2e(H - D)}{D^2 - e^2} = \frac{-4e\lambda d}{F^2 - e^2d^2}. \tag{8}$$

Lemma 1. *For ordered unit complex vertices satisfying* (8)*, $\mathcal{T}(Z) = 2vS(Z)$. No simplicity or nonzero-area assumption is needed.*

Proof. For unit U, V the division-free identity

$$[U, V](U + V) = -i(1 + \operatorname{Re}(\overline{U}V))(V - U)$$

follows by multiplication. The constant part telescopes. Write $Z_i = X_i + iY_i$. The real part of $\sum_i(X_i + X_{i+1})(Z_{i+1} - Z_i)$ telescopes as $\sum_i(X_{i+1}^2 - X_i^2) = 0$. Its imaginary part is $\sum_i[Z_i, Z_{i+1}]$, because

$$(X_i + X_{i+1})(Y_{i+1} - Y_i) = [Z_i, Z_{i+1}] + X_{i+1}Y_{i+1} - X_iY_i.$$

Multiplication by $-iv$ proves the assertion. □

Write $S_Z = S(Z)$ and $\mathcal{T}_Z = \mathcal{T}(Z)$. The origin-area identity is

$$S(0) = \frac{D}{2}S_Z. \tag{9}$$

Indeed multiply (7) by $\sin\Delta_i$ and use $\cos(\psi_i + \psi_{i+1})\sin\Delta_i = (\sin 2\psi_{i+1} - \sin 2\psi_i)/2$; the latter terms telescope. By (5), $S_Z \neq 0$ when $d \neq 0$.

4 The pedal area and full first moment

Projection onto $n_i \cdot x = h_i$ gives $Q_i = M + (h_i - M \cdot n_i)n_i$. Put $t = \overline{z}/2$ and $w_i = h_i n_i$. In complex notation,

$$Q_i = z/2 + V_i, \qquad V_i = w_i - tZ_i. \tag{10}$$

Opposite indices negate w_i and fix Z_i; hence terms odd in the w's cancel in cyclic sums. Area is translation invariant, so

$$S(M) = S(0) + |t|^2 S_Z = (D/2 + |z|^2/4)S_Z, \tag{11}$$

which proves (2).

We give the complete cubic reduction to make the essential calculation checkable. In $\mathcal{T}(V)$, the pure-w and quadratic-in-t terms cancel by antipodal pairing. The cubic term is $-|t|^2 t\mathcal{T}_Z$. For one edge put $x = n_i$, $y = n_{i+1}$, $h = h_i$, $k = h_{i+1}$, $Z = x^2$, $W = y^2$. Average the edge moment with its antipodal edge moment. In $2i$ times that average the coefficients of t and $\bar{t}$ are, respectively,

$$\begin{aligned} c &= k^2x^2 - h^2y^2 + 2hk(x^3/y - y^3/x), \\ b &= h^2x^2/y^2 - k^2y^2/x^2 + hk(x/y - y/x). \end{aligned}$$

Substitute

$$h^2 = H + \tfrac{e}{2}(x^2 + x^{-2}), \quad k^2 = H + \tfrac{e}{2}(y^2 + y^{-2}), \quad hk = \tfrac{D}{2}(y/x + x/y) + \tfrac{e}{2}(xy + (xy)^{-1}).$$

With $r = Z/W - W/Z$ and $s = Z^2/W - W^2/Z$, multiplication gives the exact edge identities

$$\begin{aligned} c &= Ds + \tfrac{3e}{2}r + (D + H)(Z - W) + e(Z^2 - W^2), \\ b &= (D/2 + H)r + \tfrac{e}{2}s + \tfrac{e}{2}(Z - W) + e(W^{-1} - Z^{-1}). \end{aligned}$$

The endpoint differences telescope. Moreover

$$\sum r = -4iS_Z, \qquad s + 2i[Z, W](Z + W) = W - Z, \qquad \sum s = -2i\mathcal{T}_Z = -4ivS_Z.$$

Consequently the two linear terms together with the cubic term give

$$\mathcal{T}(V) = -2S_Z\{t(Dv + 3e/2 + v|t|^2) + \bar{t}(D/2 + H + ev/2)\}. \tag{12}$$

These identities involve no division by $S(M)$. For an arbitrary ordered polygon, translation by c changes $\mathcal{T}$ by $6cS$. This follows by expanding $[V_i + c, V_{i+1} + c]$ and telescoping the coordinate-square and coordinate-product endpoint differences. Divide (12) by $6S(M)$, translate by $z/2$ and use (11) to obtain

$$C = \frac{z}{2} - \frac{z(D + 2H + ev) + \overline{z}(2Dv + 3e + v|z|^2/2)}{3(2D + |z|^2)}.$$

Multiplying numerator and denominator by d proves (3).

5 Definedness and an exact star example

Positivity of $S(0)$ and F makes (2) an exact classification of the zero-area locus. For a simple winding-one family with $N \geq 6$, the first $N/2$ normals occupy a half-turn in circular order. Their squares are at least three distinct circularly ordered unit-circle points. The full Z-polygon traverses their convex polygon twice, so $S_Z > 0$. Equation (9) implies $D > 0$, hence $d > 0$. Period four was treated above. This completes Theorem 1.

The negative-d case occurs in an exact primitive star. Take

$$a^2 = 221/96, \qquad b^2 = 27625/20736, \qquad \lambda = 125/96.$$

The caustic semiaxes are 1 and 25/144. Multiply the following normalized pairs coordinatewise by (a, b), retaining their order:

$$(0,1),\ (-12/13,-5/13),\ (1,0),\ (-12/13,5/13),$$
$$(0,-1),\ (12/13,5/13),\ (-1,0),\ (12/13,-5/13).$$

The verification package checks exactly that every chord is tangent to the caustic and obeys the specular reflection law. The eight vertices are distinct, and an oriented-ray crossing count gives winding three. This example was previously used in the repository's problem 5100065; no novelty is claimed for its construction. Here

$$d = -120/221, \qquad F = 2795/1728, \qquad v = 7/13, \qquad -2F/d = 123539/20736.$$

For $M = (X, Y)$, direct Cartesian projection and signed shoelace give

$$S(M) = -\frac{\sqrt{30}}{134784}(20736(X^2+Y^2) - 123539),$$
$$\mathcal{T}_x(M) = -\frac{\sqrt{30}X}{109512}(41472(X^2+Y^2) - 272935),$$
$$\mathcal{T}_y(M) = -\frac{\sqrt{30}Y}{97344}(52992(X^2+Y^2) - 347633).$$

At $M = (13\sqrt{731}/144, 0)$, $S = 0$ but $\mathcal{T} = (221\sqrt{21930}/10368, 0) \neq 0$. Thus even a removable-zero interpretation cannot supply a centroid at this diagnostic point.

Verification and disclosure

The accompanying source and verification archive provides exact symbolic checks of the displayed edge identities, unrestricted Cartesian star polynomials and a separately derived real-coordinate certificate. Independent orbit generators supply high-precision numerical controls, including stars, large periods and near-degenerate caustics. Initial floating-point failures and their precision repairs are recorded. Numerical tests corroborate, but do not replace, the all-period proof above.

AI tools were used extensively in solving, verifying, literature research, writing and adversarial review. Independent AI reviews are not external human peer review or proof-assistant formalization. This is an unrefereed research note; it has not received external human peer review. The bounded priority reports, assumptions and reproducibility instructions accompany the note. No claim of exclusive historical priority is made.

References

[1] D. Reznik, R. Garcia and J. Koiller, *Fifty new invariants of N-periodics in the elliptic billiard*, Arnold Math. J. **7** (2021), 341–355. doi:10.1007/s40598-021-00174-y. See also *Eighty new invariants*, arXiv:2004.12497v11, Table 4.

[2] M. Bialy and S. Tabachnikov, *Dan Reznik's identities and more*, Eur. J. Math. **8** (2022), 1341–1354. doi:10.1007/s40879-020-00428-7.

[3] R. E. Schwartz and S. Tabachnikov, *Centers of mass of Poncelet polygons, 200 years after*, Math. Intelligencer **38** (2016), no. 2, 29–34. arXiv:1607.04766.