The outer center-pedal area product in elliptic billiards

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16 October 2026

Abstract

For a periodic billiard family between strictly nested confocal ellipses, form the outer tangent polygon and project the ellipse center onto its side lines. We prove constancy of the product of their signed areas when the period is not divisible by four, including coprime stars and repeated traversals. This proves the assertion labelled $k_{303,b}$ in the 2020 invariant list and its 2021 published edition. The actual center-foot map is a fixed linear image of the edge-vector polygon of a phase-shifted billiard. Explicit area formulas reduce the assertion to a classical odd cyclic Jacobi identity, also giving the invariant's value. The same full theorem is stated and proved in Ferudun's contemporary version 1.1; no novelty, historical independence or priority claim is made here. A source and verification package accompanies this unrefereed, extensively AI-assisted proof and verification note.

1 The precise invariant

For an ordered closed polygon $W = (W_0, \dots, W_{N-1})$ define

$$\mathcal{A}(W) = \frac{1}{2} \sum_{j=0}^{N-1} \det(W_j, W_{j+1}), \qquad W_N = W_0. \tag{1}$$

Thus stars retain trajectory order and signed shoelace area. Let

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0, \qquad C : \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad a^2 - \alpha^2 = b^2 - \beta^2 = \lambda \in (0, b^2).$$

For a billiard orbit P_j on E tangent to C, let ℓ_j be the tangent to E at P_j, let $R_j = \ell_j \cap \ell_{j+1}$, and let Q_j be the perpendicular foot of $O = (0,0)$ on ℓ_j. The line ℓ_j is the side line $R_{j-1}R_j$ of the outer polygon. Write $A' = \mathcal{A}(R)$ and $A'_O = \mathcal{A}(Q)$.

Theorem 1. *Suppose the fixed pair (E, C) supports a periodic billiard family of least period $N \geq 3$, with $N \not\equiv 0 \pmod 4$. Then $A'A'_O$ is independent of the initial vertex. The assertion includes every admissible coprime star winding. It also holds for a listed traversal length not divisible by four that repeats a shorter orbit, with the product multiplied by the square of the repetition count.*

This is $k_{303,b}$ of Reznik–Garcia–Koiller [1, Table 4, p. 6] and [2, Table 4, p. 346]; the construction and signed-area convention are in their Sections 2 and 3.4. The conjecture first appeared in their 2020 arXiv source; the companion journal edition appeared in 2021. The source concerns a confocal

pair of ellipses. Our theorem excludes hyperbolic, focal and collapsed caustics. Feet are on the full boundary tangent lines, not on the original billiard chords. Areas of spatially sorted vertices or filled star lobes are different quantities.

The exact full theorem, with these parity classes, is also proved in Ferudun [9, Theorem 2 and Section 6], version 1.1. That version, DOI 10.5281/zenodo.23088516, is distinguished from version 1.0, which gives the primitive even arbitrary-point result and hence its centered even subset. Section 4 records the scope of the priority evidence. Here we give the explicit edge-vector and classical-identity reduction, rather than assert a first resolution.

2 Area formulas in canonical coordinates

All Jacobi functions below have modulus k, not parameter k^2. Set

$$k = \frac{\sqrt{\alpha^2 - \beta^2}}{\alpha}, \quad k' = \frac{\beta}{\alpha}, \quad K = K(k), \quad K' = K(k').$$

Stachel's canonical parametrization [3, Theorem 4.3, equation (4.9), p. 1614] gives, after choosing orientation,

$$P_j = P(u + 2jv), \quad P(z) = (-a\,\mathrm{sn}\,z, b\,\mathrm{cn}\,z), \tag{2}$$
$$v = \frac{2\tau K}{N} \in (0, K), \quad \gcd(\tau, N) = 1, \quad 0 < \tau < N/2, \qquad a = \frac{\alpha\,\mathrm{dn}\,v}{\mathrm{cn}\,v}, \quad b = \frac{\beta}{\mathrm{cn}\,v}.$$

Only windings supported by the fixed pair are asserted. Put

$$f(t) = \frac{\mathrm{sn}\,t\,\mathrm{cn}\,t}{\mathrm{dn}\,t}, \qquad T(z) = \sum_{j=0}^{N-1} \mathrm{dn}(z + 2jv).$$

Cyclic reindexing gives $T(z + 2v) = T(z)$. Jacobi addition [10, Section 22.8] gives, for $H(z) = (-A\,\mathrm{sn}\,z, B\,\mathrm{cn}\,z)$,

$$\det(H(z - t), H(z + t)) = \frac{2AB\,\mathrm{sn}\,t\,\mathrm{cn}\,t\,\mathrm{dn}\,z}{1 - k^2\,\mathrm{sn}^2\,z\,\mathrm{sn}^2\,t} = ABf(t)\big(\mathrm{dn}(z - t) + \mathrm{dn}(z + t)\big). \tag{3}$$

Summing consecutive determinants proves $\mathcal{A}(P) = abf(v)T(u)$. For any indexed polygon, determinant expansion also gives

$$\mathcal{A}((P_{j+1} - P_j)_j) = 2\mathcal{A}(P) - \frac{1}{2}\sum_j \det(P_j, P_{j+2}).$$

Applying (3) with $t = 2v$ and reindexing the sums yields

$$\mathcal{A}((P_{j+1} - P_j)_j) = ab\big(2f(v) - f(2v)\big)T(u). \tag{4}$$

This identity does not require stepping by two to form a single cycle.

The side P_jP_{j+1} is tangent to C at $B(u + (2j + 1)v)$, where $B(z) = (-\alpha\,\mathrm{sn}\,z, \beta\,\mathrm{cn}\,z)$. Indeed both $P(z \pm v)$ satisfy $-\mathrm{sn}(z)X/\alpha + \mathrm{cn}(z)Y/\beta = 1$ by addition formulas. Its pole with respect to E is R_j, so elementary conic polarity gives

$$R_j = \mathrm{diag}(a^2/\alpha^2, b^2/\beta^2)B(u + (2j + 1)v) = \mathrm{diag}(a/\alpha, b/\beta)P(u + (2j + 1)v).$$

Taking the fixed linear area scale,

$$A'(u) = \frac{a^2b^2}{\alpha\beta} f(v)T(u + v). \tag{5}$$

The intersections R_j are finite: parallel endpoint tangents would force antipodal endpoints, whose center-crossing chord cannot touch the strictly interior ellipse C.

The actual perpendicular foot

The tangent at $P(z)$ has equation $n(z)\cdot X = 1$, where $n(z) = (-\operatorname{sn} z/a, \operatorname{cn} z/b)$. Thus

$$Q(z) = \frac{n(z)}{n(z)\cdot n(z)} = \frac{(-ab^2 \operatorname{sn} z, a^2 b \operatorname{cn} z)}{a^2 - (a^2-b^2)\operatorname{sn}^2 z}. \tag{6}$$

Set $w = K - v \in (0, K)$. Quarter-period formulas give

$$\operatorname{sn} w = \frac{\operatorname{cn} v}{\operatorname{dn} v}, \qquad \operatorname{cn} w = \frac{k' \operatorname{sn} v}{\operatorname{dn} v} > 0, \qquad \operatorname{dn} w = \frac{b}{a}.$$

The sum formulas for $P(z+w) + P(z-w)$ have denominator

$$D(z) = 1 - k^2 \operatorname{sn}^2 z \operatorname{sn}^2 w = \frac{a^2 - (a^2-b^2)\operatorname{sn}^2 z}{a^2}$$

and numerator $(-2a \operatorname{sn} z \operatorname{cn} w \operatorname{dn} w, 2b \operatorname{cn} z \operatorname{cn} w)$. Using $P(z-2K) = -P(z)$ now proves the pointwise identity

$$Q(z) = L\big(P(z+K-v) - P(z+K+v)\big), \quad L = \frac{\operatorname{diag}(b/a, 1)}{2 \operatorname{cn} w}, \quad \det L = \frac{b}{4a \operatorname{cn}^2 w}. \tag{7}$$

The global minus sign relative to the edge vector preserves area in two dimensions. Therefore (4) proves

$$A_O'(u) = \frac{b^2}{4 \operatorname{cn}^2(K-v)} \big(2f(v) - f(2v)\big) T(u+K-v). \tag{8}$$

All real denominators used above are positive or nonzero. No polygon area has been divided out.

The underlying geometry is classical. With e_j the forward unit chord velocity and $J = \sqrt{\lambda}/(ab)$ the Joachimsthal integral, reflection gives $Q_j = (e_{j-1} - e_j)/(2J)$, as follows directly from [6, proof of Theorem 4] and in the center-foot telescope of [7, Theorem 4.1]. The skew hodograph $\operatorname{diag}(a,b)e_j$ is described in [6, Lemma 3.1 and Remark 3.2]. In the present coordinates its exact phase is $P(u + (2j+1)v + K)$. Equation (7) verifies this mechanism directly for the same caustic and fixes its normalization.

3 The classical trace identity and proof of the theorem

Lemma 2. *Let q be positive and odd, and $0 < k < 1$. Define*

$$U_q(z) = \sum_{j=0}^{q-1} \operatorname{dn}(z + 2Kj/q), \qquad C_q = \prod_{j=1}^{(q-1)/2} \operatorname{cs}^2(2Kj/q), \quad \operatorname{cs} = \operatorname{cn}/\operatorname{sn}.$$

The empty product is one. Then, as a meromorphic identity,

$$U_q(z) U_q(z+K) = \frac{(k')^q}{C_q^2}. \tag{9}$$

Proof. Khare–Lakshminarayan–Sukhatme [4, Appendix A, equation (86) in arXiv v1; (85) in v2] give the odd cyclic product-to-sum identity

$$\prod_{j=0}^{q-1} \operatorname{dn}(z + 2Kj/q) = C_q U_q(z).$$

Multiply it by its K-translate and use $\operatorname{dn}(t+K) = k'/\operatorname{dn} t$. Meromorphic continuation handles isolated zeros and poles. For $q = 1$ this is just the quarter-period formula. □

This trace relation is prior mathematics, also connected to the generalized Landen formulas in [5]. For completeness its product-to-sum precursor can be checked directly. The product $V(z)$ and sum $U_q(z)$ are $2K$-periodic and $2\mathrm{i}K'$-antiperiodic. Their poles are simple at the shifts of $\mathrm{i}K'$ by the real subgroup $H = \{2Kj/q\}$ modulo $2K$. At $\mathrm{i}K'$, the standard formulas $\operatorname{Res} \mathrm{dn} = -\mathrm{i}$ and $\mathrm{dn}(\mathrm{i}K' + t) = -\mathrm{i}\,\mathrm{cs}\,t$ give

$$\frac{\operatorname{Res}_{\mathrm{i}K'} V}{\operatorname{Res}_{\mathrm{i}K'} U_q} = (-\mathrm{i})^{q-1} \prod_{j=1}^{q-1} \mathrm{cs}(2Kj/q) = C_q,$$

since $\mathrm{cs}(2K - t) = -\mathrm{cs}\,t$. The same ratio holds at every pole by subgroup periodicity and imaginary antiperiodicity. Thus $V - C_q U_q$ is entire elliptic, hence constant, and its antiperiodicity forces zero. This is a reproof of the classical identity, not a new special-function assertion. The meromorphic pole-cancellation method itself is established, for example in [4, 6, 8].

Proof of Theorem 1. First consider a primitive orbit and put $q = N/\gcd(N, 2)$, $r = N/q$. Coprimality in (2) means that the shifts $2jv$ modulo $2K$ exhaust H exactly r times. Consequently $T(z) = rU_q(z)$. The hypothesis $N \not\equiv 0 \pmod 4$ says precisely that q is odd. Because T has period $2v$, $T(u + K - v) = T(u + K + v)$. Multiplying (5) and (8), then using Lemma 2, gives the explicit value

$$A'A'_O = \frac{a^2 b^4 f(v)\big(2f(v) - f(2v)\big)}{4\alpha\beta\,\mathrm{cn}^2(K - v)} \, \frac{r^2 (k')^q}{C_q^2}, \tag{10}$$

independent of u. These determinant and subgroup computations apply to stars without reordering. Reversing traversal changes both area signs and preserves their product. A d-fold traversal repeats each shoelace sum d times. If its listed length is not divisible by four, neither is its least period, so the primitive theorem applies and its product is multiplied by d^2. □

The coefficient in (10) has units length to the fourth power, as required. Nondegenerate one- or two-bounce orbits are impossible here: the two-bounce segment passes through O and cannot touch C. The circular limit is separately elementary: all fixed-step regular-star families differ by rigid rotation, so both signed areas are constant. There is no assertion for a general period divisible by four. For example, on $x^2/4 + y^2 = 1$ the diamond with vertices $(2, 0), (0, 1), (-2, 0), (0, -1)$ and the rectangle with vertices $(\pm 4/\sqrt{5}, \pm 1/\sqrt{5})$ in boundary order are billiards with the same caustic $\lambda = 4/5$. Their outer/pedal products are respectively $8 \cdot 4 = 32$ and $10 \cdot 5 = 50$. Thus the omitted parity cannot simply be dropped.

4 Attribution, evidence and limitations

The note proves the originally conjectured invariant in the stated domain. It does not establish that this is a new theorem or the first proof. Ferudun's version 1.1 [9] contains the exact full result. Its Zenodo record was created on 1 October 2026 at 18:12:36 UTC; version 1.0 was created at 16:56:14 UTC. GitHub server check-suite timestamps place receipt of this note's unchanged candidate proof tree at 05:35:19 UTC that day; PR 209 was created at 05:48:56 UTC. These observations do not certify past public repository visibility, exclude an earlier website posting, or establish historical independent non-derivation. The bounded priority audit and its byte-bound chronology are included in the supplement. They leave historical ordering and novelty undetermined. The old trace identities, foot/velocity and hodograph mechanisms, canonical coordinates and meromorphic method are expressly credited above. Chavez-Caliz's even original/outer area product [8, Theorem

6] concerns a different pair of areas; its auxiliary $2q$-family also yields trace constancy, as detailed in the supplement.

The verification package preserves the original candidate and inherited review unchanged, three fresh mathematical/source audits and two priority audits, derivations, original and new exact controls, replay receipts and failure corrections. Exact symbolic calculations and arbitrary-precision samples support the analytic proof; finite samples are not universal proofs or interval-certified computations. In particular the period-four examples check the actual reflection and common-caustic hypotheses. The package states software versions and how to rerun the mathematical checks without network access; source retrieval and release operations are separate from those checks. Third-party primary papers and local software environments are not redistributed.

AI use and review status. AI tools were used extensively for discovery, derivation, programming, literature searches, writing and adversarial checking, including separately prompted automated reviewers. This is an unrefereed preprint. No independent human peer review or formal proof verification is claimed. Automated reviews and exact controls do not constitute either. Alec Kriebel is the named author; the preserved audit records state the actual scope of the checks.

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