Original-ball paths in Teichmüller space: an elementary corollary of Lenzhen–Rafi

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13 October 2026 Version 1.0 Unrefereed expository note

Abstract

We record an elementary path corollary of Lenzhen–Rafi's established uniform quasiconvexity theorem. For each fixed closed oriented surface of genus at least two, every pair of points in any closed Teichmüller ball can be joined inside that same ball by a continuous rectifiable path whose length is at most their ambient distance plus a topology-dependent constant. Radially moving the endpoints inward before joining them gives the deduction. This supplies an affirmative derivation of Farb's Question 3.5 from prior Teichmüller geometry. We make no claim to historical priority or a new substantive Teichmüller theorem.

1 The original-ball conclusion

Fix a closed oriented surface S of genus $g \geq 2$. Its marked Teichmüller space $\mathcal{T}(S)$ has the metric $d_T(X,Y) = \frac{1}{2}\log K(X,Y)$, where $K(X,Y)$ is the least quasiconformal dilatation in the marking class. Balls below are closed balls for this metric, and lengths are ordinary metric lengths of paths.

Theorem 1. *There is a finite constant $c_g \geq 0$ such that, for every center o, every finite radius $R \geq 0$, and every $x, y \in B_{d_T}(o,R)$, there is a continuous rectifiable path in $B_{d_T}(o,R)$ joining x to y with length at most $d_T(x,y) + 4c_g$.*

The same c_g works for all centers, radii and endpoints at the fixed genus. In particular, ambient distance at most k gives a path of length at most $N_g(k) = k + 4c_g$. Taking $k = 2$ satisfies Farb's definition of almost convexity and Question 3.5, which asks for bounded-length paths inside the original radius ball [1, pp. 24–25]. No numerical value, genus-uniform bound or optimal constant is asserted.

The substantive geometric input is Lenzhen–Rafi's published uniform ball theorem [2, Theorems C and 17]. It allows a minimizing segment to leave the endpoint-radius ball by a bounded additive amount. The short general deduction below converts that allowance into a path in the original ball. It requires no thickness condition and no replacement of the Teichmüller metric.

2 The elementary metric deduction

Lemma 2. *Let (M,d) be a geodesic metric space. Suppose one finite $c \geq 0$ has the following property: for every $o, a, b \in M$, some minimizing segment from a to b satisfies*

$$d(o,z) \leq \max\{d(o,a), d(o,b)\} + c \quad \text{at every point } z \text{ on that segment.} \tag{1}$$

Then every $x, y \in B(o,R)$ can be joined by a continuous rectifiable path in $B(o,R)$ of length at most $d(x,y)+4c$.

Proof. If $R < c$, join x to o and o to y by radial minimizing segments. Their concatenation lies in $B(o,R)$ and has length at most $2R$, which is at most $d(x,y) + 4c$.

Suppose $R \geq c$. On radial minimizing segments from x and y toward o, move inward by

$$\alpha = \min\{c, d(o,x)\}, \qquad \beta = \min\{c, d(o,y)\},$$

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obtaining a and b. If an original endpoint had radial distance below c, its new point is o; otherwise its radius has decreased by c. Thus $d(o,a), d(o,b) \le R - c$. Choose the a-to-b segment in (1); all of it lies in $B(o,R)$. The two radial end pieces also lie in that ball. The resulting concatenation has length

$$\alpha + d(a,b) + \beta \le d(x,y) + 2\alpha + 2\beta \le d(x,y) + 4c.$$

Arclength parametrizations of the three pieces paste to give an actual continuous rectifiable path. Constant pieces cover $a = b$, coincident endpoints and endpoints at the center. The case $R = c$ is included; so are $c = 0$ and $R = 0$. No attainment of an intrinsic-distance infimum, properness, or continuous choice of segments is needed. □

3 Application of the established theorem

Lenzhen–Rafi's extremal-length result [2, Theorem 15] gives a finite $K_g \ge 1$, depending only on the topology, such that

$$\mathrm{Ext}_z(\gamma) \le K_g \max\{\mathrm{Ext}_a(\gamma), \mathrm{Ext}_b(\gamma)\} \tag{2}$$

for every essential simple closed curve γ and every point z on the minimizing Teichmüller segment from a to b. The constant is uniform in the curve, endpoints and intermediate point. Their Theorem 17 proves the corresponding radial ball statement. For clarity, it can also be obtained directly from Kerckhoff's formula [3], in the normalization recorded in [4, Section 2, equation (2.2)]:

$$d_T(o,w) = \frac{1}{2} \sup_\gamma \log \frac{\mathrm{Ext}_w(\gamma)}{\mathrm{Ext}_o(\gamma)}.$$

Divide (2) by the positive $\mathrm{Ext}_o(\gamma)$ and take the supremum of half logarithms. This gives

$$d_T(o,z) \le \max\{d_T(o,a), d_T(o,b)\} + \frac{1}{2}\log K_g.$$

There is no need to select a curve or foliation attaining the supremum. Minimizing Teichmüller segments exist between every pair of points; see also [4, Section 2]. Lemma 2, with $c_g = \frac{1}{2}\log K_g$, proves Theorem 1.

The original published input and these geodesic and distance-formula facts are established external results. This note does not reprove their foundational Teichmüller theory. The journal's extremal-length and ball theorem numbers are respectively 15 and 17; the earlier arXiv:1002.3939v1 uses 13 and 15 for those body theorems.

4 Attribution, scope and disclosure

All substantive Teichmüller estimates here are due to Lenzhen–Rafi. The contribution of this exposition is to write out the radial-retreat implication with the exact original-ball and path-length quantifiers. Historical priority for that elementary implication or this corollary has not been established. The bounded source audit located no explicit full-scope predecessor in the inspected bodies; it does not establish worldwide absence or that Farb's question remained open in 2026.

Fortier Bourque–Rafi's nonconvexity result [4] concerns the direct minimizing segment and is compatible with the concatenated path above. Petyt–Zalloum's Theorem I [5], present in their 2024 preprint and 2026 published version, gives almost convexity with $N(k) = k+6$ in a dual wall space. Its map from the original space is quasiisometric; it does not identify original Teichmüller balls and path lengths. We make no comparison of numerical constants across those different metrics. Neither the lemma nor this note asserts optimality of $4c_g$ in Teichmüller space.

AI tools were used extensively in deriving the corollary, verifying the proof, adversarial review, source auditing and preparation. This is an unrefereed expository preprint. Its AI adversarial audits do not constitute conventional human peer review, journal acceptance or formal proof-assistant verification. Supporting authored audit records and source identities are provided separately; third-party paper bodies are not redistributed.

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