Verification of the odd-period focal inversion area product

Alec Kriebel

Independent researcher

ORCID: 0009-0001-9320-500X

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Abstract

For an odd periodic billiard in a proper confocal ellipse pair, invert its original vertices in unit circles centered at the original outer foci. The product of the two signed straight-edge areas is independent of phase. This is the historical invariant k903,a of Reznik–Garcia–Koiller and Conjecture 1 of Reznik–Garcia–Helman. Ferudun's 2026 preprint proves the same theorem using substantially the same meromorphic divisor mechanism. This attributed verification note gives a self-contained intrinsic flag-curve presentation: four isotropic common tangents identify every possible pole, odd torsion separates their orbit grids, and reversal supplies the opposite-focus zeros needed for compactness. Primitive admitted star windings, signs, and indexed repetitions are specified. Four exact finite billiard controls accompany the proof, including two same-caustic even orbits with different products. No novelty or absolute-priority claim is made.

1 Historical target and attribution

Reznik, Garcia, and Koiller define signed polygon area in [1, Eq. (1)], original-vertex inversion about the original billiard foci in [1, §3.9], and the odd focal-inverse area product in [1, Table 10, p. 12, k903,a]. The row is dated 10/20. The shorter journal edition does not contain this row. Reznik, Garcia, and Helman give the triangle product as Proposition 7 and its odd-period extension as Conjecture 1 in [2, p. 6].

Ferudun's *Products of the Two Focal Inversion Areas in Elliptic Billiards*, version 1.0 [3], states and proves this same strict odd-period theorem, including admitted stars, reversal, and repetitions. Its Jacobi-coordinate proof and the flag-curve proof below share the essential architecture: bound the area poles, use reversal to reduce their orders, obtain opposite-focus zeros, and apply compactness. The intrinsic formulation here is an attributed verification and exposition of that theorem and mechanism.

Complexification, flag involutions, and removal of poles on a compact curve are established tools for Poncelet invariants; see Akopyan–Schwartz–Tabachnikov [4], Chávez-Caliz [5], and Schwartz [6]. Their cited invariants concern other observables. For the present inverse areas the focal divisor and adjacent-pole calculation must still be supplied; these are made explicit below. Source and chronology limitations are recorded in Section 7.

2 Statement and conventions

Fix

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad C : \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \qquad \begin{array}{c} a > b > 0, \quad 0 < \lambda < b^2, \\ \alpha^2 = a^2 - \lambda, \quad \beta^2 = b^2 - \lambda. \end{array} \tag{1}$$

Put $c = \sqrt{a^2 - b^2} > 0$, so $\alpha^2 - \beta^2 = c^2$, and let $F_f = (f, 0)$, $f = \pm c$. For an ordered real billiard $(P_0, \ldots, P_{N-1})$, with indices taken modulo N, define

$$I_f(P) = F_f + \frac{P - F_f}{\|P - F_f\|^2}, \qquad A_f = \frac{1}{2} \sum_{j=0}^{N-1} \det(I_f(P_j), I_f(P_{j+1})). \tag{2}$$

Only the vertices are inverted; successive image vertices are joined by straight segments. For $P = (a\cos\theta, b\sin\theta)$, $\|P - F_f\| = a - f\cos\theta \geq a - c > 0$, so real inversion is nonsingular.

Theorem 1. *Suppose the pair (1) admits a primitive real billiard of odd period $N \geq 3$. For each oriented family of such billiards, the product $A_c A_{-c}$ in (2) is independent of phase and is strictly positive. The assertion includes each coprime star winding actually admitted by the pair. Reversal negates both areas and preserves the product. Repeating a primitive cycle $r \geq 1$ times multiplies each area by r and the product by r^2.*

Primitive means that the ordered billiard flag first returns after N steps. The star formulation does not permit arbitrary reordering of one vertex set. In Stachel's convention [7, Theorem 4.3], the modulus is $k = c/\alpha \in (0, 1)$, the vertices are $(-a\,\mathrm{sn}\,u, b\,\mathrm{cn}\,u)$, and the step is $2v$, where

$$\mathrm{cn}\,v = \beta/b, \qquad a = \alpha\,\mathrm{dn}\,v/\,\mathrm{cn}\,v, \qquad v = 2\nu K(k)/N, \quad 1 \leq \nu < N/2, \quad \gcd(\nu, N) = 1.$$

The software parameter for these functions is k^2. These relations specify the period and winding conventions; the proof uses no complex Jacobi period identity. Circular, coincident, degenerate, and hyperbolic caustics are outside the strict statement.

3 The compact flag curve

Let X be the normalized projective curve of flags (P, ℓ), where $P \in E$ and ℓ is a tangent to C through P. Projection $\pi : X \to E$ has degree two. Its branch vertices are $E \cap C$:

$$x^2 = \frac{a^2\alpha^2}{c^2}, \qquad y^2 = -\frac{b^2\beta^2}{c^2}. \tag{3}$$

They are four distinct finite points with nonzero coordinates. The conic gradients there are independent, since $a^2/\alpha^2 \neq b^2/\beta^2$.

An explicit check on normalization and infinity is useful. Parametrize E by

$$x = a\frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2bt}{1 + t^2}.$$

The tangent discriminant gives the double cover, up to a nonzero square factor, as

$$q^2 = \beta^2 t^4 + (4\alpha^2 - 2\beta^2)t^2 + \beta^2. \tag{4}$$

The discriminant in t^2 is $16\alpha^2 c^2 \neq 0$, and the constant and leading coefficients are nonzero. Thus the four branch roots are distinct and nonzero. At $t = \pm\mathrm{i}$, the right side is $-4c^2 \neq 0$; these are

the geometric points at infinity of E. The parameter value $t = \infty$ instead lies over $(-a, 0)$ and is unbranched. The compact cover is connected, because it has simple branch points, and has genus one by Riemann–Hurwitz.

Let σ exchange the two outer endpoints on ℓ, and let τ exchange the two caustic tangents through P. These extend to involutions of X, with

$$T = \tau\sigma, \quad \pi\tau = \pi, \quad \pi T = \pi\sigma, \qquad \sigma T \sigma = T^{-1}. \tag{5}$$

The map T advances along the tangent chord and then chooses the other outgoing caustic tangent: it is the billiard successor. Each involution has fixed points and is nonidentity. A fixed-point involution on a genus-one curve acts in an Abel coordinate as $u \mapsto d-u$: its linear part is -1, since a nonidentity translation cannot fix a point. Consequently T is a translation. A primitive period-N real flag makes its translation element have exact order N on all of X.

A fixed flag of σ corresponds to a common tangent of the two conics. For a line $ux+vy+w_0 = 0$, the two tangency conditions are

$$a^2u^2 + b^2v^2 = w_0^2, \qquad \alpha^2u^2 + \beta^2v^2 = w_0^2.$$

Their difference is $\lambda(u^2 + v^2) = 0$. Thus the four common tangents, all distinct and transverse as dual-conic intersections, are

$$z = x + \mathrm{i}y = \pm c, \qquad w = x - \mathrm{i}y = \pm c. \tag{6}$$

Write their σ-fixed flags as $s_{f,z}, s_{f,w}$, $f = \pm c$.

4 Complete focal pole inventory

Translation of every image vertex by $-F_f$ leaves its closed shoelace area unchanged. In the two complex coordinates, inversion then has the meromorphic form

$$Z_f = \frac{1}{w - f}, \qquad W_f = \frac{1}{z - f}. \tag{7}$$

These are algebraic extensions of ordinary real inversion; complex conjugation is not being used as a holomorphic operation. On E,

$$z - f = -\frac{a + f}{1 + t^2}\left(t - \frac{\mathrm{i}b}{a + f}\right)^2. \tag{8}$$

Hence W_f has a double pole at the isotropic contact

$$P_{f,z} = \left(\frac{a^2}{f}, \frac{\mathrm{i}b^2}{f}\right), \qquad w - f = \frac{2b^2}{f} \neq 0.$$

At $P_{f,w} = (a^2/f, -\mathrm{i}b^2/f)$, the roles of Z_f, W_f are reversed. Each contact has caustic residual

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 = -\frac{\lambda^2}{\alpha^2\beta^2} \neq 0.$$

Therefore π is unramified there. Above $P_{f,z}$ the two flags are $s_{f,z}$ and $\tau s_{f,z} = T s_{f,z}$; each pole has order two. The analogous statement holds above $P_{f,w}$.

There are no hidden poles at infinity. At either geometric infinity, $y = \pm\mathrm{i}(b/a)x$, so both z and w have simple poles with nonzero leading coefficients proportional to $a \pm b$. Their reciprocals in (7) vanish. We have thus listed every polar flag of both inverse coordinates.

Lemma 1. *If N is odd, the four grids $G_s = \{T^j s : j \in \mathbb{Z}/N\}$, for the four fixed flags in (6), are pairwise disjoint.*

Proof. If $s' = T^j s$ and both flags are fixed by σ, then (5) gives $s' = T^{-j} s$. Hence $T^{2j} s = s$. Exact order N and oddness give $j = 0 \pmod N$, so $s' = s$. □

Every possible area pole lies on one of these grids, because its coordinate poles occur at s, Ts. The proof of the lemma uses no primality or generic-parameter assumption. It covers composite odd periods and every admitted primitive winding.

5 Reversal removes every product pole

For a phase $Q \in X$, put $R_j = W_f(T^j Q)$, $S_j = Z_f(T^j Q)$. The complexified signed area is

$$\mathcal{A}_f(Q) = \frac{1}{4\mathrm{i}} \sum_j (R_j S_{j+1} - S_j R_{j+1}) = \frac{1}{4\mathrm{i}} \sum_j R_j (S_{j+1} - S_{j-1}). \tag{9}$$

It is invariant under T. Moreover $\pi T^j \sigma Q = \pi T^{1-j} Q$, so σ reverses the vertex list and

$$\mathcal{A}_f(\sigma Q) = -\mathcal{A}_f(Q). \tag{10}$$

At $s = s_{f,z}$, only vertices 0 and 1 have poles, both of order two in the *same* coordinate R. Every S_j is regular there by Lemma 1. In particular, the determinant for the edge joining the two singular vertices contains no $R_0 R_1$ term. The area pole therefore has order at most two before any symmetry is applied.

Choose a local coordinate h at the fixed point s with $\sigma : h \mapsto -h$. Equation (10) eliminates the double coefficient and constant term, giving $\mathcal{A}_f = \rho/h + O(h)$. For an explicit principal-part check, write

$$R_0 = r_{-2} h^{-2} + r_{-1} h^{-1} + O(1), \qquad D_0 = S_1 - S_{-1} = d_0 + d_1 h + O(h^2).$$

Reversal gives $R_1(h) = R_0(-h)$ and $D_1(h) = S_2 - S_0 = -D_0(-h)$. Thus the polar terms in (9) are $[R_0(h) D_0(h) - R_0(-h) D_0(-h)]/(4\mathrm{i})$, and

$$\rho = \frac{r_{-2} d_1 + r_{-1} d_0}{2\mathrm{i}}. \tag{11}$$

No unsupported trace-cancellation identity is needed.

At the same phase the opposite-focus area $\mathcal{A}_{-f}$ is regular, since its pole grids are disjoint from this one. It is odd by (10), so $\mathcal{A}_{-f} = b_1 h + O(h^3)$. The product is removable and takes value ρb_1. At $s_{f,w}$ interchange R, S; the same pole bound and oddness apply. Translation handles all points of each grid. Elsewhere the inverse coordinates are regular. The product is therefore holomorphic on compact connected X, and is constant. This proves phase constancy in Theorem 1.

Orient real tangent chords with C on their left. Each focus is strictly inside C, since $\alpha^2 - c^2 = \beta^2 > 0$. Consequently

$$\det(P_j - F_f, P_{j+1} - F_f) > 0.$$

The corresponding determinant after inversion is this positive number divided by $\|P_j - F_f\|^2 \|P_{j+1} - F_f\|^2$. Both signed areas, and hence their constant product K_0, are positive. This argument is edgewise and applies to stars. Reversal negates both shoelace sums. Repetition simply repeats the sum r times, proving the stated r, r^2 laws.

Positivity also sharpens the divisor description without circularity. If $\rho = 0$ at any listed grid, both factors would be analytic and odd there and their product would be zero. Thus all these poles

and opposite zeros are exactly simple; there are no additional zeros. As reduced sums of N points per grid,

$$(\mathcal{A}_f) = G_{s_{-f,z}} + G_{s_{-f,w}} - G_{s_{f,z}} - G_{s_{f,w}}.$$

6 Exact finite controls and the parity restriction

Take $a^2 = 8$, $b^2 = 5$, so $f = \pm\sqrt{3}$. For $\lambda = 40/9$, two positive primitive triangles are

$$H = ((2\sqrt{2}, 0), (-4\sqrt{2}/3, 5/3), (-4\sqrt{2}/3, -5/3)),$$
$$V = ((0, \sqrt{5}), (-8/3, -\sqrt{5}/3), (8/3, -\sqrt{5}/3)).$$

Direct inversion and shoelace sums give

	A_c	A_{-c}	A_cA_{-c}
H	$\sqrt{2}/10 - \sqrt{3}/80$	$\sqrt{2}/10 + \sqrt{3}/80$	$5/256$
V	$\sqrt{5}/16$	$\sqrt{5}/16$	$5/256$

The source formula comparison requires care. On p. 3 of [2], the printed $\delta = \sqrt{a^4 - a^2b^2 - b^4}$ has radicand -1 at these axes. Replacing its final minus by plus gives $\delta = 7$ and makes the Proposition 7 expression agree with $5/256$ at this exact control. This checks that control only; no general corrected formula or inspected journal correction is asserted. The universal proof above uses neither expression.

Oddness is essential. For arbitrary $a > b > 0$, put $S = a^2 + b^2$, $\lambda = a^2b^2/S$, so $\alpha = a^2/\sqrt{S}$, $\beta = b^2/\sqrt{S}$. Two genuine primitive four-period billiards for this same pair are

$$D = ((a, 0), (0, b), (-a, 0), (0, -b)), \qquad R = ((\alpha, \beta), (-\alpha, \beta), (-\alpha, -\beta), (\alpha, -\beta)).$$

For D, its side lines $\pm x/a \pm y/b = 1$ are tangent to C because $\alpha^2/a^2 + \beta^2/b^2 = 1$; reflection holds by axis symmetry. For R, the sides $x = \pm\alpha, y = \pm\beta$ are tangent, its vertices lie on E, and the outer normal is proportional to $(\pm 1, \pm 1)$, which reflects its horizontal and vertical edges. Four distinct vertices establish primitivity.

For either focus, direct inversion gives $A_f(D) = 2/(ab)$ and $A_f(R) = 1/(\alpha\beta)$. Their products differ:

$$A_c(D)A_{-c}(D) = \frac{4}{a^2b^2}, \qquad A_c(R)A_{-c}(R) = \frac{S^2}{a^4b^4}, \qquad \frac{S^2 - 4a^2b^2}{a^4b^4} = \frac{c^4}{a^4b^4} > 0.$$

At the specified axes and $\lambda = 40/13$, the rectangle vertices are $(\pm 8/\sqrt{13}, \pm 5/\sqrt{13})$ in cyclic order. The two products are $1/10$ and $169/1600$. These even controls were derived during the present verification and carry no discovery claim.

The accompanying standard-library exact verifier checks the four polygons' incidences, segment tangencies, reflections, inversions, areas, reversal, and triple repetition. It also distinguishes inversion radius scaling from physical similarity: radius 2 multiplies each area by 16 and the product by 256; scaling the orbit and both foci by $s = 3/2$, with radius fixed at 1, multiplies area by s^{-2} and product by s^{-4}. Its 245 checks in 27 nonempty groups are finite controls, not a certificate of the all-period proof. Explicit exceptions remain active under Python optimization; deliberate false and malformed inputs are tested separately. All four finite controls are convex winding-one trajectories; higher stars are covered by the analytic proof, not by this finite suite.

7 Scope, record dates, and assistance

The priority comparison was bounded. In particular the official 15-page preview of [8] lists §8.5, "Invariant area product," but omits that section's text; the full section was unavailable for this comparison. The RGH journal full text and its typography were not inspected. These access gaps do not establish absence of earlier proofs. The direct exact theorem and overlapping mechanism in [3] are expressly credited.

The preserved GitHub PR206 record reports creation at 2026-10-01T05:36:26Z; its preserved HEAD has committer time 2026-10-01T05:36:15Z. Ferudun's Zenodo record reports creation at 2026-10-02T07:08:59.065762+00:00, and the paper declares 2 October 2026 as its date. These are facts about those records. They do not certify first public availability of every byte, visibility history, intellectual independence, literal simultaneity, or absolute priority. The note makes no claim that the historical conjecture was still open when this verification was prepared.

AI tools were used extensively in research, proof development, drafting, computation, and verification. Separate AI adversarial checks are not independent human peer review. This is an unrefereed research note; no conventional human refereeing or proof-assistant certification is claimed.

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