The odd-period focal-pedal sum invariant

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Abstract

We verify the focal-pedal invariant $k601$ of Reznik, Garcia and Koiller: for an odd-period billiard in a fixed proper confocal ellipse pair, the product of the sums of the distances from the two foci to the original side lines is constant. The proof combines Stachel's canonical billiard coordinates with the classical odd-order generalized Landen formulas of Khare and Sukhatme. We give the required trace normalization directly from convergent classical Fourier series. Admissible primitive winding classes and odd repetitions are included. An exact four-period example shows why parity matters. This is an attributed verification, without a novelty or priority claim.

1 Statement and conventions

Let $E : x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$. Fix $0 < \lambda < b^2$ and the proper nested confocal elliptic caustic

$$C : \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \qquad \alpha^2 = a^2 - \lambda, \quad \beta^2 = b^2 - \lambda, \qquad c^2 = a^2 - b^2 = \alpha^2 - \beta^2.$$

The common foci are $F_\pm = (\pm c, 0)$. For a closed billiard $P_0, \dots, P_{N-1}$, let L_i be the complete supporting line of P_iP_{i+1}, with indices modulo N, and put

$$q_{\pm,i} = \operatorname{dist}(F_\pm, L_i), \qquad I(P) = \left(\sum_{i=0}^{N-1} q_{+,i}\right)\left(\sum_{i=0}^{N-1} q_{-,i}\right).$$

These are ordinary Euclidean perpendicular distances; the foot is not clipped to the segment. Set

$$k = \frac{c}{\alpha}, \quad m = k^2, \quad k' = \frac{\beta}{\alpha} = \sqrt{1-m}, \qquad K = K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}.$$

Our $\operatorname{sn}(u \mid m), \operatorname{cn}(u \mid m), \operatorname{dn}(u \mid m)$ use the *parameter* m. References using modulus k, including DLMF, use $k = \sqrt{m}$.

Theorem 1. *Suppose the family with outer ellipse E and caustic C has odd physical least period $N \geq 3$. Then $I(P)$ is independent of the initial vertex, and*

$$I(P) = \beta^2 \left[\sum_{j=0}^{N-1} \frac{1}{\operatorname{dn}(4Kj/N \mid m)}\right]^2. \tag{1}$$

Every admissible primitive winding $0 < \tau < N/2$ with $\gcd(\tau, N) = 1$, including stars, and reversal of orientation are covered. Repeating an odd primitive orbit r times multiplies I by r^2; in particular, an odd indexed repetition has this constancy.

The target is $k601$ in Table 7, p. 9, of the exact preprint edition [2], and Table 7, p. 349, of the published article [1]. Section 3.7 defines the distances from the foci to the pedal feet on the original sides. We make the standard complete-line pedal interpretation explicit. The row itself does not literally print a primitive-period or complete-supporting-line qualifier; odd repetitions reduce harmlessly by the r^2 law. The question mark in the historical tables does not certify present-day openness.

2 A classical odd-order trace formula

The entire special-function step is a direct consequence of earlier generalized Landen formulas, not a new identity. The shared parameter is specified in [7, Eq. (53)]; the common-argument odd-order formulas are in its Section 3.1, pp. 12–13, Eqs. (68)–(70), with normalizations in Eqs. (14), (32). All-odd pair-correlation constancies were already stated in [5, Table 1, Eq. (29)]; a general proof framework and explicit correlations appear in [6, Sections 1–2 and Appendix A]. We verify the needed normalization from classical Fourier series.

Lemma 2 (Classical odd-order trace formula)**.** *For $0 < m < 1$ and odd $N \geq 3$, define*

$$D(t) = \sum_{j=0}^{N-1} \operatorname{dn}(t + 4Kj/N \mid m), \qquad H(t) = \sum_{j=0}^{N-1} \operatorname{cn}(t + 4Kj/N \mid m).$$

Let

$$q = e^{-\pi K(1-m)/K(m)}, \quad Q = q^N, \quad M = \left[\frac{\theta_2(0,Q)}{\theta_3(0,Q)}\right]^4, \quad K_M = K(M), \quad \gamma = \frac{K}{NK_M}.$$

Then $0 < M < 1$, $\gamma > 0$, and for real t,

$$D(t) = \gamma^{-1} \operatorname{dn}(t/\gamma \mid M), \qquad H(t) = \frac{\sqrt{M}}{\sqrt{m}\,\gamma} \operatorname{cn}(t/\gamma \mid M). \tag{2}$$

In particular,

$$D(t)^2 - mH(t)^2 = \frac{1-M}{\gamma^2}. \tag{3}$$

Proof. The classical Fourier expansions [8, 22.11.2–3] are

$$\operatorname{dn}(t \mid m) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n\geq 1} \frac{q^n}{1+q^{2n}} \cos\frac{n\pi t}{K},$$

$$\operatorname{cn}(t \mid m) = \frac{2\pi}{\sqrt{m}\,K} \sum_{n\geq 0} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos\frac{(2n+1)\pi t}{2K}.$$

They converge absolutely and uniformly for real t, since $0 < q < 1$. Thus finite grid sums may be interchanged with the series. The root-of-unity sum $\sum_{j=0}^{N-1} e^{2\pi i \ell j/N}$ is N if $N \mid \ell$ and zero otherwise. For dn it keeps $N \mid 2n$, equivalently $N \mid n$ because N is odd. For cn it keeps $2n+1 = N(2s+1)$. Consequently

$$D(t) = \frac{N\pi}{2K} + \frac{2N\pi}{K} \sum_{\ell\geq 1} \frac{Q^\ell}{1+Q^{2\ell}} \cos\frac{N\ell\pi t}{K},$$

$$H(t) = \frac{2N\pi}{\sqrt{m}\,K} \sum_{s\geq 0} \frac{Q^{s+1/2}}{1+Q^{2s+1}} \cos\frac{N(2s+1)\pi t}{2K}.$$

The standard nome inversion formulas [8, 22.2.1–2] identify M as the parameter with nome Q and give $K_M = (\pi/2)\theta_3(0,Q)^2$. Comparing the series with those of parameter M, using $\gamma K_M = K/N$, proves (2). Finally, $\mathrm{dn}^2(z \mid M) - M\,\mathrm{cn}^2(z \mid M) = 1 - M$ gives (3). □

For comparison with [7], its α is our γ. Its α_1, denoted here by η, satisfies

$$\gamma = \left[\sum_{j=0}^{N-1} \mathrm{dn}(2Kj/N \mid m)\right]^{-1}, \qquad \eta = \left[\sum_{j=0}^{N-1} \mathrm{cn}(4Kj/N \mid m)\right]^{-1}, \qquad M = m\gamma^2/\eta^2.$$

The dn grids agree because dn has period $2K$ and 2 is invertible modulo odd N. The cn sum at zero is positive by its filtered Fourier series. These relations specify the shared parameter and argument without conflating the unequal cn and dn amplitudes.

3 The geometric bridge

We use Stachel's canonical parametrization [3, Theorem 4.3 and Eq. (4.9)].[1] For an admissible primitive winding τ, put

$$h = \frac{2\tau K}{N}, \qquad a = \frac{\alpha\,\mathrm{dn}(h \mid m)}{\mathrm{cn}(h \mid m)}, \qquad b = \frac{\beta}{\mathrm{cn}(h \mid m)}.$$

Then $0 < h < K$, and the family has vertices

$$P_i = (-a\,\mathrm{sn}(u + 2ih \mid m),\, b\,\mathrm{cn}(u + 2ih \mid m)), \qquad u \in \mathbb{R}.$$

This is a credited geometric input, not a new proof of integrability.

For $v_i = u + (2i+1)h$, the chord line is

$$-\frac{\mathrm{sn}(v_i \mid m)}{\alpha} X + \frac{\mathrm{cn}(v_i \mid m)}{\beta} Y = 1. \tag{4}$$

To check the contact phase directly, use the difference-addition identity

$$\mathrm{cn}\,x\ \mathrm{cn}\,y + \mathrm{dn}(y-x)\,\mathrm{sn}\,x\ \mathrm{sn}\,y = \mathrm{cn}(y-x),$$

where the common parameter is m. Substitution of either vertex parameter $v_i - h$ or $v_i + h$ into (4) yields 1. The identity follows by solving the cn/dn difference-addition system; its real determinant $1 - m\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y$ is at least $1 - m > 0$. Equation (4) is tangent to C at $(-\alpha\,\mathrm{sn}\,v_i, \beta\,\mathrm{cn}\,v_i)$.

The normal norm is

$$\sqrt{\frac{\mathrm{sn}^2 v_i}{\alpha^2} + \frac{\mathrm{cn}^2 v_i}{\beta^2}} = \frac{\mathrm{dn}\,v_i}{\beta}.$$

The ordinary distances are therefore

$$q_{+,i} = \beta\frac{1 + k\,\mathrm{sn}\,v_i}{\mathrm{dn}\,v_i}, \qquad q_{-,i} = \beta\frac{1 - k\,\mathrm{sn}\,v_i}{\mathrm{dn}\,v_i}. \tag{5}$$

Here $\mathrm{dn}\,v_i \geq k' > 0$ and $1 \pm k\,\mathrm{sn}\,v_i \geq 1 - k > 0$, so the absolute values in Euclidean distance are resolved by explicit positive bounds. The same focal-line support geometry and the pointwise

[1] The alternating sign printed for the dn half-period shift on p. 1613 is a typo: $\mathrm{dn}(u + 2K) = \mathrm{dn}\,u$. We use the correct period and quarter-period identities from DLMF.

identity $q_{+,i}q_{-,i} = \beta^2$ already appear in Bialy–Tabachnikov [4, Lemma 4.2]. The pointwise product alone does not establish the product of sums.

Coprimality of τ, N makes the set of v_i modulo $4K$ exactly $\{v + 4Kj/N : 0 \le j < N\}$. With

$$A(v) = \sum_j \frac{1}{\operatorname{dn}(v + 4Kj/N)}, \qquad B(v) = \sum_j \frac{\operatorname{sn}(v + 4Kj/N)}{\operatorname{dn}(v + 4Kj/N)}$$

and common parameter m, (5) gives $I = \beta^2(A^2 - mB^2)$. The classical quarter-period identities

$$\operatorname{dn}(z + K) = \frac{k'}{\operatorname{dn} z}, \qquad \operatorname{cn}(z + K) = -\frac{k' \operatorname{sn} z}{\operatorname{dn} z}$$

give $A(v) = D(v + K)/k'$, $B(v) = -H(v + K)/k'$. Thus Lemma 2 yields

$$I(P) = \alpha^2[D(v + K)^2 - mH(v + K)^2] = \alpha^2 \frac{1 - M}{\gamma^2},$$

independent of u. At $v = 0$, the odd function $\operatorname{sn} z/\operatorname{dn} z$ pairs under negation on the grid, so $B(0) = 0$. This proves (1). Reversal permutes the same side lines. Repetition multiplies each sum by r, proving the r^2 law and Theorem 1.

4 An exact parity boundary and the circular limit

There is no unrestricted even-period extension. Consider

$$E : \frac{x^2}{3/2} + \frac{y^2}{3/4} = 1, \qquad C : x^2 + 4y^2 = 1, \qquad F_\pm = (\pm\sqrt{3}/2, 0).$$

The axis-vertex diamond

$$(0, \sqrt{3}/2), \quad (-\sqrt{3/2}, 0), \quad (0, -\sqrt{3}/2), \quad (\sqrt{3/2}, 0)$$

is a primitive four-period billiard. Its sides, written with right-hand side 1, have normal norm $\sqrt{2}$; each pair of opposite sides has distances summing to $2/\sqrt{2}$ from either focus. Each four-side sum is $2\sqrt{2}$, hence $I = 8$. The rectangle with cyclic vertices

$$(1, 1/2), \quad (-1, 1/2), \quad (-1, -1/2), \quad (1, -1/2)$$

lies on E, and its sides $x = \pm 1$, $y = \pm 1/2$ are tangent to C. Each focal distance sum is 3, hence $I = 9$. The diamond satisfies specular reflection by axial symmetry. At the rectangle vertices the ellipse normals have components $(\pm 2/3, \pm 2/3)$, giving the same reflection law. Thus these are two primitive four-period billiards in the same proper-confocal family with unequal products.

When $c = 0$, the foci coincide with the center and every side has distance β from it, so directly $I = N^2\beta^2$. No singular Landen normalization at $m = 0$ is needed. Collapsed or hyperbolic caustics, grazing limits and replacement of line distance by segment distance are outside the assertion.

5 Attribution and verification status

The source quantity and observation are due to Reznik, Garcia and Koiller; the focal support identity is credited to Bialy–Tabachnikov; the canonical coordinates are used in Stachel's form. The

complete odd-order trace identity and pair correlations are existing generalized Landen and cyclic-identity mathematics [5, 6, 7]. The proof resolves the stated invariant question by an attributed application of these inputs. We claim no theorem, formula or proof-method novelty, firstness, independent discovery or simultaneity.

The bounded prior comparison distinguishes these established inputs from an earlier explicit full geometric $k601$ application. It has not established such an earlier printed application in its selected sources, and it does not exclude one elsewhere. No present-day openness or priority conclusion follows from this bounded finding or the historical table's question mark.

AI systems were used extensively in research, proof development, drafting, adversarial review and computation. This note is unrefereed. AI review and computational checks are not conventional human peer review, a proof-assistant formalization or a formally certified proof. Numerical checks are finite, noninterval diagnostics; the universal assertion rests on the proof above.

References

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[8] National Institute of Standards and Technology, *NIST Digital Library of Mathematical Functions*, https://dlmf.nist.gov/, version 1.2.8, 15 September 2026. Chapter 22, especially Sections 22.2, 22.4–22.6, 22.8 and 22.11.