Loss of total-space positivity for the normalized relative Kähler–Ricci flow

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Abstract

We give a negative answer to the total-space positivity-preservation question for the relative Kähler–Ricci flow defined by Naumann on $\mathcal{O}_E(r)$. For the ample rank-two bundle $E = \mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$, we construct a smooth strictly positive initial metric on $\mathcal{O}_E(2) \simeq \mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2)$. For the defining flow with probability-normalized fiber measures, its horizontal curvature at one point is exactly $1/96 - (1 - e^{-2t})/24$, hence equals $-1/96$ at $t = (\log 2)/2$. The argument uses the exact evolution of a second base derivative along a stationary central fiber and an explicit global positivity certificate.

1 Question and statement

Naumann's report [1] ends on p. 2443 with the question of preserving positivity on the total space when a positive metric on $\mathcal{O}_E(1)$ is evolved through its rth power. The question remains explicit in [2, §3.3]. We answer it for the flow actually defined in [2, (13)]. The flow's construction itself was already given in those sources. The older report abbreviates the denominator without a time subscript; we use the evolving measure explicit in the published defining equation.

Let $f : X = \mathbf{P}(E^*) \to S$ use the convention $f_*\mathcal{O}_E(1) = E$. For a metric $e^{-\phi}$ on $L = \mathcal{O}_E(r)$, the isomorphism $K_{X/S}^{-1} = L \otimes f^*(\det E)^{-1}$ identifies its fiber restrictions with anticanonical densities after a determinant frame is chosen. Define

$$\partial_t \phi_t = \log \frac{\mathrm{MA}(\phi_t)}{\mu_{\phi_t}}, \qquad \mu_\phi = \frac{e^{-\phi}\, dA_z}{I(s)}, \quad I(s) = \int_{\mathbf{P}^1} e^{-\phi(s,z)}\, dA_z. \tag{1}$$

Here the fiber dimension in the example is one, and MA is curvature volume divided by its fixed total mass. The density notation uses compatible anticanonical frames; it is not the integral of a globally defined scalar function against an arbitrarily chosen background area form. A determinant frame change cancels from μ_ϕ. Addition of a base function changes neither probability measure. In local coordinates, with $g = \phi_{z\bar{z}} > 0$,

$$\partial_t \phi = \log g + \phi + \log I + C, \tag{2}$$

where C is a fixed constant determined by the area convention and the curvature mass. In particular the reaction coefficient for this weight is one. Omitting the fixed mass normalization in the report changes only a function of time in the solution and leaves its curvature unchanged.

Theorem 1. *For $S = \mathbf{P}^1$ and $E = \mathcal{O}_S(1) \oplus \mathcal{O}_S(1)$, there is a smooth Hermitian metric $e^{-\phi_0}$ on $L = \mathcal{O}_E(2)$ with strictly positive curvature on the compact total space X, such that its solution of (1) satisfies*

$$(\phi_t)_{s\bar{s}}(0,0) = \frac{1}{96} - \frac{1-e^{-2t}}{24}. \tag{3}$$

At $t_* = (\log 2)/2$ this coefficient is $-1/96$. Fiberwise curvature remains positive. The positive square-root metric on $\mathcal{O}_E(1)$ gives admissible initial data for the source's question.

We use the finite-time existence and smoothness statement of [2, Theorem 5], including smoothness on $X \times [0,\infty)$. Its proof reduces the fiberwise flow to the normalized anticanonical Kähler–Ricci flow; see also [3]. No regularity or convergence of an infinite-time limit in base directions is needed here. Uniqueness can be checked directly: for the difference v of two solutions, $|\log I(\phi) - \log I(\widetilde{\phi})| \le \|v\|_{C^0}$ on each fiber. At a fiber maximum of v the logarithmic Hessian difference is nonpositive. Applying the maximum principle to v and $-v$ gives an upper Dini derivative bound $D^+\|v\|_{C^0} \le 2\|v\|_{C^0}$. Grönwall's inequality proves uniqueness for common initial data and therefore preservation of lifted symmetries.

2 A globally positive initial metric

Write $E = \mathcal{O}_S(1) \otimes \mathbf{C}^2$. Then $X \simeq \mathbf{P}^1_s \times \mathbf{P}^1_z$ and $\mathcal{O}_E(1) = \mathcal{O}(1,1)$, so E is ample and $L = \mathcal{O}(2,2)$. Set

$$\epsilon = \frac{1}{96}, \qquad \delta = \frac{1}{4}, \qquad u = |s|^2, \quad v = |z|^2, \quad p = \frac{u}{1+u}, \quad x = \frac{v}{1+v}.$$

In the finite chart take

$$\phi_0 = 2\log(1+u) + \frac{2-\epsilon}{1+u} + 2\log(1+v) + \delta\frac{u}{1+u}\left(\frac{v}{1+v}\right)^2. \tag{4}$$

The logarithms are standard weights on $\mathcal{O}(2,2)$ and the remaining terms are global smooth functions. Explicitly, under $w = 1/s$ subtract the transition weight $2\log|s|^2$. The base terms become $2\log(1+|w|^2) + (2-\epsilon)|w|^2/(1+|w|^2)$, with $p = (1+|w|^2)^{-1}$. Under $\zeta = 1/z$ subtract $2\log|z|^2$; the fiber logarithm becomes $2\log(1+|\zeta|^2)$ and $x^2 = (1+|\zeta|^2)^{-2}$. These formulas also show smoothness in the joint infinity chart.

In product Fubini–Study unit frames the Hessian diagonal coefficients and squared mixed modulus are

$$\begin{aligned} A &= (1+u)^2(\phi_0)_{s\bar{s}} = 4p + (\epsilon + \delta x^2)(1-2p), \\ B &= (1+v)^2(\phi_0)_{z\bar{z}} = 2 + 2\delta p x(2-3x), \\ Q &= (1+u)^2(1+v)^2|(\phi_0)_{s\bar{z}}|^2 = 4\delta^2 p(1-p)x^3(1-x). \end{aligned} \tag{5}$$

Direct differentiation proves these identities. They extend continuously to the closed square $0 \le p, x \le 1$, covering every infinity point. Throughout this square,

$$A \ge \epsilon + \frac{167}{48}p =: A_0, \qquad B \ge \frac{3}{2} =: B_0, \qquad Q \le \frac{p}{4} =: Q_0.$$

For explicit certificates,

$$\begin{aligned} A - A_0 &= \delta x^2 + 2\delta p(1-x^2) \ge 0, \\ B - B_0 &= 2\delta\{(1-p) + p(1-x)(1+3x)\} \ge 0, \\ Q_0 - Q &= 4\delta^2 p\{1 - (1-p)x^3(1-x)\} \ge 0. \end{aligned}$$

Because A_0, B_0 are positive,

$$AB - Q \geq A_0 B_0 - Q_0 = \frac{1}{64} + \frac{159}{32} p > 0. \tag{6}$$

Thus the curvature is strictly positive in every direction everywhere. This continuum certificate does not rely on sampled points. Dividing the weight by two gives the required positive root metric.

3 An exact second-variation equation

Rotational symmetry in the base, preserved by uniqueness, gives on $s = 0$

$$(\phi_t)_s = (\phi_t)_{\bar{s}} = (\phi_t)_{s\bar{z}} = 0. \tag{7}$$

The initial central-fiber weight is $2\log(1+|z|^2)+2-\epsilon$. Its two probability measures in (1) coincide, so uniqueness gives

$$\phi_t(0,z) = 2\log(1+|z|^2) + 2 - \epsilon \quad \text{for every finite } t \geq 0.$$

Define

$$w(z,t) = (\phi_t)_{s\bar{s}}(0,z), \qquad g_F = \frac{2}{(1+|z|^2)^2}, \qquad \Delta_F = \frac{(1+|z|^2)^2}{2}\partial_z \partial_{\bar{z}}.$$

The sign convention makes Δ_F nonpositive on nonconstant eigenfunctions. Differentiating (2), using (7), yields $(\log g)_{s\bar{s}} = g_F^{-1} w_{z\bar{z}} = \Delta_F w$. Smoothness and compactness justify differentiation under the fiber integral:

$$I_s = 0, \qquad I_{s\bar{s}} = -I \int_{\mathbf{P}^1} w \, \mu_F, \qquad (\log I)_{s\bar{s}} = -\int_{\mathbf{P}^1} w \, \mu_F.$$

Consequently the second derivative of the actual nonlinear solution obeys the closed equation

$$w_t = \Delta_F w + w - \int_{\mathbf{P}^1} w \, \mu_F, \qquad w(z,0) = \epsilon + \delta x^2. \tag{8}$$

This is an exact identity, not a linearization approximation. The geodesic curvature is $c(\phi_t) = (\phi_t)_{s\bar{s}} - |(\phi_t)_{s\bar{z}}|^2/(\phi_t)_{z\bar{z}}$; by (7) it equals w on the central fiber.

4 Solution and strict negativity

The pushforward of μ_F under $x = v/(1+v)$ is Lebesgue measure on $[0,1]$, since radial integration gives $dv/(1+v)^2 = dx$. For a smooth $F(x)$,

$$\Delta_F F = \tfrac{1}{2}\{x(1-x)F'' + (1-2x)F'\}.$$

Let $P_1 = 2x-1$ and $P_2 = 6x^2 - 6x + 1$. Both have zero mean; $\Delta_F P_1 = -P_1$, $\Delta_F P_2 = -3P_2$, and $x^2 = 1/3 + P_1/2 + P_2/6$. Hence (8) has solution

$$w(x,t) = \epsilon + \frac{\delta}{3} + \frac{\delta}{2} P_1(x) + \frac{\delta}{6} e^{-2t} P_2(x). \tag{9}$$

To justify its identification with the actual second derivative, the difference of two solutions of (8) with common initial data has zero mean: integrating its equation shows that the mean is constant. The difference then solves $a_t = \Delta_F a + a$; the maximum principle applied to $e^{-t}a$ proves uniqueness.

At $z = 0$, $P_1 = -1$ and $P_2 = 1$, so $w(0,t) = \epsilon - \delta(1 - e^{-2t})/6$, which is (3). For $t = t_*$ it equals $-1/96$, and is negative for every $t > \frac{1}{2}\log(4/3)$. Because the mixed Hessian vanishes there, this is negative curvature on the actual tangent vector $\partial/\partial s$. All initial data were strictly positive on a compact total space, and t_* is finite. This proves Theorem 1.

5 Normalization, scope, and verification

There is a convention discrepancy between the defining equation [2, (13)] and a literal same-weight/time reading of the displayed rc term in [2, (14)]. A calibration independent of our example is the positive product weight $\phi_{\mathrm{prod}} = 2\log(1+|s|^2) + 2\log(1+|z|^2)$. It is stationary under (1). At $s = 0$ its geodesic curvature is $c = 2$, the fiber Laplacian and Kodaira–Spencer terms vanish, and $(\log I)_{s\bar{s}} = -2$. A same-convention rc term with $r = 2$ would therefore give $c_t = 2$, contradicting stationarity. Our proof uses (1) and derives (8) directly. We do not assign an editorial explanation or undocumented intention to the source. For comparison, setting $\psi = \phi/r$ and $\tau = t/r$ transforms (2) into

$$\psi_\tau = \log \psi_{z\bar{z}} + r\psi + \log \int e^{-r\psi}\, dA_z + C',$$

so a coefficient r can occur with the corresponding weight and time changes. Such positive rescaling leaves the occurrence of positivity loss unchanged. Probability normalization is essential: omitting $\log I$ changes the flow and the second-variation equation.

The result resolves the source's universal positivity-preservation question for its defining flow. It does not settle the Griffiths conjecture, exclude special initial metrics, or address other normalizations. The example's bundle itself already has Griffiths-positive metrics. The prior construction of the flow is not claimed as a new contribution.

The accompanying verification package contains the manuscript source, exact symbolic checks, analytic derivations, source and priority audit, and adversarial review records. Symbolic computations check the algebra and global inequality certificates; smooth parabolic existence is the cited analytic input, not a computational conclusion. Historical first priority is not asserted; the accompanying audit records the scope and limits of the literature search. In particular, Berman's preservation theorem concerns a nonnormalized weight flow, and his earlier [3, Remark 4.11] already identifies normalized anticanonical negativity for effectively parametrized Fano families. Those statements do not supply this locally isotrivial projective-space bundle example on $\mathcal{O}_E(r)$. They preclude a broad claim of first positivity loss for normalized relative flows. A relevant Naumann habilitation listed by his university in 2025 was identified, but its text was not recovered; this is a specific unresolved priority lead recorded in the accompanying audit.

AI use and review status. AI tools were used extensively in developing, writing, reproducing, and adversarially checking this result. Internal adversarial reviews were performed by AI agents. This is an unrefereed preprint and has not received human peer review; these internal checks do not constitute such review.

References

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