

Nonunique nearest stable complex quadratics

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Abstract

The closed set of complex monic quadratics with roots in the closed left half-plane is not prox-regular at z^2 . For every fixed positive-definite real inner product on coefficient space, unstable quadratics with multiple attained global nearest stable quadratics occur arbitrarily close to z^2 . The conclusion also holds in the full exact-degree quadratic coefficient space. A direct proof exhibits a limiting normal whose cubic support dominates a quartic squared displacement. The same obstruction disproves set prox-regularity of the whole abscissa epigraph. In contrast, the monic quadratic abscissa is function prox-regular at the origin relative to each finite limiting subgradient. Thus the historical question about local unique stable-polynomial projection has a negative answer for complex coefficients, while this example does not refute the separate function condition.

1 Question, conventions, and result

Lewis [2, §8, pp. 166–168] asks whether the stable monic set is prox-regular, after specifying complex coefficients and their real Euclidean coefficient metric. The later Oberwolfach question [3, p. 106, Question 1] asks whether every degree- k polynomial near z^k has a unique nearest stable polynomial. That short question does not specify its coefficient field, normalization, or metric. We address its complex coefficient convention, which is explicit in the earlier set question and in the cited variational theory [4, 5]. No conclusion for the real-only general question is asserted.

For $p_{a,b}(z) = z^2 + az + b$, identify $(a, b) \in \mathbb{C}^2$ with $\mathbb{R}^4$, in the order $(\operatorname{Re} a, \operatorname{Im} a, \operatorname{Re} b, \operatorname{Im} b)$. Let $\alpha(p)$ be the maximum real part of a root, and set

$$S = \{(a, b) : \alpha(p_{a,b}) \leq 0\}, \quad c_0 = (0, 0).$$

Imaginary-axis roots, including multiple roots, are allowed. For a fixed real symmetric positive-definite matrix G , write $\langle u, v \rangle_G = u^T G v$ and $\|u\|_G^2 = u^T G u$. In the full coefficient space put

$$S_{\text{full}} = \{(\ell, a, b) \in \mathbb{C}^3 : \ell \neq 0, \alpha(\ell z^2 + az + b) \leq 0\}, \quad c_{\text{full}} = (1, 0, 0).$$

A nearest point always refers to a global minimum over the stated stable set.

Theorem 1. *For every fixed positive-definite real coefficient inner product, S is not prox-regular at c_0 , and every neighborhood of c_0 contains an unstable quadratic with at least two distinct attained global nearest points in S . The corresponding statements hold for S_{full} at c_{full} . The abscissa epigraph is not prox-regular as a set at height zero over either base point, for every fixed positive-definite real inner product on the respective coefficient/height space.*

The first-order polynomial root and normal calculus and the connection between prox-regularity and projection are classical [4, 2, 6]. The underlying stability-boundary cusp is a positive Whitney-umbrella geometry; the related real-quartic boundary was studied by Bottema [1] and is reviewed as a Whitney umbrella by Kirillov–Overton [7]. We give the coefficient normal, metric obstruction, and global attainment arguments directly. This note makes no assertion of firstness or of the question’s present-day openness. The accompanying priority memorandum records the primary sources checked and the remaining inaccessible bodies. Nearest-stable coefficient approximation was already studied by Moses–Liu [10] for real disk-stable polynomials, including recognition of possible global nonuniqueness, and by Orban de Xivry [11, Chapter 4] for complex half-plane coefficients. Our assertion concerns attained ties in every neighborhood of the specified quadratic base, with all fixed coefficient inner products; generic nonconvexity or competing local algorithmic minima alone do not establish that assertion.

2 A limiting normal that is not proximal

For a locally closed set C in a finite-dimensional inner product space, a vector v is a proximal normal at $x \in C$ if, for some $\tau > 0$, x is a nearest point to $x + \tau v$; equivalently, a local quadratic upper support can be used and the normal step made sufficiently small. Limits of proximal normals at convergent points are limiting normals. Set prox-regularity at x_0 requires a uniform local projection property near each limiting normal at x_0 [6, Definition 1.1 and Proposition 1.2]. In particular, every limiting normal at x_0 must itself be proximal there. For closed sets in finite dimension, set prox-regularity at x_0 is equivalent to single-valued nearest projection on some neighborhood of x_0 [6, Theorem 1.3(a),(k)]. These facts apply in any fixed inner product, by a linear isometry to Euclidean space. Locally closed sets may be clipped inside a closed ball, with the base point in its interior.

For $t > 0$ take the stable path

$$p_t(z) = z^2 + t^2 z + t^2 + it^3 = (z + it)(z + t^2 - it), \quad c_t = (t^2, 0, t^2, t^3). \quad (1)$$

The root $\lambda = -it$ is simple, with the other root strictly to its left. Thus α is smooth near c_t . Implicit differentiation in all complex coefficient directions gives

$$d\lambda = -\frac{\lambda da + db}{t^2 - 2it}, \quad \nabla \alpha(c_t) = \gamma_t n_t, \quad \gamma_t = \frac{2}{t(t^2 + 4)}, \quad n_t = (-t, -t^2/2, -t/2, 1).$$

Here n_t is a real support covector. In metric G its representing normal vector is $G^{-1}n_t$.

At each fixed t , Taylor’s theorem for the local smooth constraint $\alpha \leq 0$ yields $n_t^T(c - c_t) \leq K_t \|c - c_t\|_G^2$ for nearby stable c . Expanding squared distances shows that, for a sufficiently small positive τ , c_t is the unique closest such point to $c_t + \tau G^{-1}n_t$. Choose the step smaller also than half the radius of that local metric ball; the triangle inequality excludes all points outside the ball. Consequently $G^{-1}n_t$ is an actual proximal normal to S , without requiring a uniform K_t or a uniform root gap.

Let $n_0 = (0, 0, 0, 1)$. Since $c_t \rightarrow c_0$ and $n_t \rightarrow n_0$, $G^{-1}n_0$ is a limiting normal at c_0 . However,

$$n_0^T c_t = t^3, \quad \|c_t\|^2 = 2t^4 + t^6. \quad (2)$$

More generally, writing $e = (1, 0, 1, 0)$ and $f = (0, 0, 0, 1)$,

$$\|c_t\|_G^2 = t^4(e^T G e + 2te^T G f + t^2 f^T G f), \quad e^T G e > 0.$$

For every $\tau > 0$,

$$\|c_t - \tau G^{-1}n_0\|_G^2 - \|\tau G^{-1}n_0\|_G^2 = \|c_t\|_G^2 - 2\tau t^3 < 0$$

when t is sufficiently small. The competitors converge to the base point. Thus $G^{-1}n_0$ is not even a local proximal normal at c_0 , disproving prox-regularity for every fixed G .

The continuous quadratic formula

$$\alpha(a, b) = -\frac{\operatorname{Re} a}{2} + \frac{1}{2} \sqrt{\frac{|a^2 - 4b| + \operatorname{Re}(a^2 - 4b)}{2}} \quad (3)$$

shows that S is closed. It is nonempty and all distances to it are attained: a minimizing sequence is bounded by comparison with c_0 , and has a convergent subsequence. The projection characterization therefore gives non-single-valued projections in every neighborhood of c_0 , with at least two distinct minimizers. Their inputs are unstable, since a stable input has only itself as a distance-zero nearest point. This proves the monic assertion.

Remark 2 (Boundary chart and real coefficients). Writing $a = x + iy, b = u + iv$ and substituting $z = w - iy/2$ gives $w^2 + xw + U + iV$, where $U = u + y^2/4$ and $V = v - xy/2$. The quadratic formula gives the exact closed stability chart

$$x \geq 0, \quad U \geq 0, \quad V^2 \leq x^2 U.$$

Indeed its largest root real part is $(-x + \sqrt{(|x^2 - 4U - 4iV| + x^2 - 4U)/2})/2$. Stability is equivalent to $x \geq 0$ and $|x^2 - 4U - 4iV| \leq x^2 + 4U$. Squaring with its right side nonnegative gives precisely the displayed inequalities, including $x = 0$ and $U = 0$. The path (1) lies on the upper smooth sheet $V = x\sqrt{U}$. This is the positive umbrella solid discussed above. For real monic quadratics the stability set instead is the convex orthant $a \geq 0, b \geq 0$; projection is unique for every fixed inner product. The imaginary constant coefficient in (1) is essential to this example.

3 Full degree and the epigraph

In coordinate order $(\operatorname{Re} \ell, \operatorname{Im} \ell, \operatorname{Re} a, \operatorname{Im} a, \operatorname{Re} b, \operatorname{Im} b)$, direct differentiation at the same monic points gives

$$d\lambda = -\frac{\lambda^2 d\ell + \lambda da + db}{t^2 - 2it}, \quad \nabla \alpha(1, t^2, t^2 + it^3) = \gamma_t N_t,$$

$$N_t = (t^3/2, -t^2, -t, -t^2/2, -t/2, 1) \longrightarrow N_0 = (0, 0, 0, 0, 0, 1).$$

Both leading-coefficient directions are included. The earlier argument makes $G^{-1}N_t$ proximal locally for each full coefficient metric G . The displacement from c_{full} has zero leading coordinates, support t^3 , and a squared metric norm of order t^4 with a positive leading coefficient. Hence the limiting normal $G^{-1}N_0$ is not proximal.

The exact-degree stable set is not globally closed, so attainment requires localization. On $\ell \neq 0$ the abscissa equals $\alpha(a/\ell, b/\ell)$ and is continuous. Choose a closed metric ball B of radius R about c_{full} with $\ell \neq 0$ throughout. Then $C = S_{\text{full}} \cap B$ is nonempty and compact. Its local normals along the path are unchanged, so C fails prox-regularity at the base. For an input q with $\|q - c_{\text{full}}\|_G < R/3$, every stable point outside B is more than $2R/3$ from q , while the stable base is less than $R/3$ away. Boundary points of B cannot minimize either. Thus the global nearest points in C and S_{full} agree. Applying the closed-set projection characterization to C proves the full-degree assertion of Theorem 1, including global attainment and unstable inputs.

At $(c_t, 0)$ the epigraph has smooth constraint $\alpha(c) - r \leq 0$. Its outward proximal normal covectors can be scaled to

$$v_t = (n_t, -1/\gamma_t) \longrightarrow (n_0, 0).$$

For full degree replace n_t by N_t . For any fixed positive-definite coefficient/height metric H , their representing proximal normal vectors are $H^{-1}v_t$. Comparison with the epigraph points $(c_t, 0)$ again gives support t^3 and squared distance of order t^4 . The limit is not proximal at height zero, proving the epigraph assertion. The monic epigraph is closed; in full degree a small closed ball inside $\ell \neq 0$ supplies local closedness. The limiting support covector is horizontal. Its representing vector is horizontal for a coefficient/height product metric, and can tilt when the metric mixes those coordinates.

4 Why the function condition is different

Function prox-regularity at a point relative to a finite limiting subgradient v_0 localizes the subgradients near v_0 as well as the base points and function values. Its epigraph correspondence is at the normal $(v_0, -1)$ [8, Definition 1.1 and p. 205]. The preceding horizontal normal does not test that condition: the unscaled gradients along (1) diverge. The distinction is substantive, as the next proposition shows.

Proposition 3. *The monic complex quadratic abscissa (3) is function prox-regular at $(0, 0)$ relative to every finite limiting subgradient, in the real Euclidean coefficient metric. The constants may depend on the specified subgradient.*

Proof. Put $D = a^2 - 4b$, $L = (-\infty, 0] \subset \mathbb{C}$, and $\Phi(D) = \sqrt{(|D| + \operatorname{Re} D)/2}$. Real pairings on $\mathbb{C}$ are understood. For $D \notin L$, Φ is smooth and $\|\nabla \Phi(D)\| = 1/(2\sqrt{|D|})$. At $D = -s < 0$, rationalizing the square root gives

$$\Phi(D + h) = \frac{|\operatorname{Im} h|}{2\sqrt{s}} + O(|h|^2),$$

so its regular, proximal, and limiting subdifferentials are the imaginary interval $\{iq : |q| \leq 1/(2\sqrt{s})\}$. The limiting statement follows also by the two off-ray gradient limits at that fixed point.

For $|D| \leq \delta$ one has

$$\Phi(D) \geq \frac{\operatorname{dist}(D, L)}{2\sqrt{\delta}}. \quad (4)$$

For $\operatorname{Re} D \geq 0$, use $\operatorname{dist}(D, L) = |D|$ and $\Phi(D) \geq \sqrt{|D|/2}$. For $\operatorname{Re} D < 0$, use

$$\Phi(D) = \frac{|\operatorname{Im} D|}{\sqrt{2(|D| - \operatorname{Re} D)}}.$$

At zero every finite q in the convex normal cone $N_L(0) = \{q : \operatorname{Re} q \geq 0\}$ has local linear lower support by (4), after choosing δ sufficiently small. Conversely, increments on the negative real ray force $\operatorname{Re} q \geq 0$. Off-ray gradients diverge in norm as $D \rightarrow 0$, while bounded on-ray subgradients belong to $N_L(0)$. Hence all three subdifferentials at zero equal $N_L(0)$, with the limiting conclusion valid also when defined using limits of proximal subgradients.

The global real smooth diffeomorphism $(a, b) \mapsto (a, a^2 - 4b)$, with inverse $b = (a^2 - D)/4$, transports these subdifferentials exactly. First-order lower supports transform by the Jacobian; quadratic proximal supports transform by Taylor's theorem, and limits follow by continuity of the inverse Jacobian. In particular any limiting subgradient v of α has

$$v = -\frac{1}{2}e_{\operatorname{Re} \alpha} + \frac{1}{2}J_D(a, b)^T q, \quad v_b = -2q, \quad q \in \partial \Phi(D). \quad (5)$$

This is a direct special case of the standard smooth composition calculus [8, Theorem 3.1].

Fix a finite $v_0 \in \partial\alpha(0, 0)$. In a fixed neighborhood of v_0 , (5) bounds $|q| \leq M$, for some $M \geq 1$. Restrict the coefficient ball so $|D| < \delta = 1/(16M^2)$ throughout. Every nearby base x with a subgradient near v_0 must have $D_x \in L$: an off-ray gradient would have norm greater than $2M$. Then $\Phi(D_x) = 0$ and $q \in N_L(D_x)$. For any comparison point y in that ball and its nearest-ray point z , convex normal support and (4) give

$$q \cdot (D_y - D_x) \leq M \operatorname{dist}(D_y, L) \leq \Phi(D_y).$$

Writing $\Delta = y - x$ and using the exact identity $D_y - D_x = J_D(x)\Delta + (\Delta a)^2$ now yields

$$\alpha(y) \geq \alpha(x) + v \cdot \Delta + \frac{1}{2}q \cdot (\Delta a)^2 \geq \alpha(x) + v \cdot \Delta - \frac{M}{2}\|\Delta\|^2.$$

With curvature constant $M + 1$ the inequality is strict for $y \neq x$, as in [8, Definition 1.1]. It holds without needing a restriction on the base function value, hence in particular on the required function-attentive neighborhood. Formula (3) supplies continuity. This proves the proposition. $\square$

Cross [9, Chapter 5] studies the finite-subgradient function condition and gives a negative example for a nonlinear two-real-parameter matrix family (Example 5.38). Its coefficient map is $(x_1, x_2) \mapsto (a = 0, b = -x_1^2 + x_2^2)$, whose derivative at the origin vanishes. That example concerns the parameter metric and does not identify distinct nearest complex coefficient polynomials. His real-quadratic pointwise proximal-subgradient calculation on pp. 131–133 is also a different statement. Neither such parameter results nor partial smoothness replace the ambient support and attainment arguments above.

Reproducibility and disclosure

The verification package contains this source, a standard-library exact rational verifier for the root, normal, and support identities, deterministic high-precision boundary controls for Proposition 3, and an authored source/priority memorandum. Finite checks support reproducibility; the limiting, universal-metric, and global projection conclusions follow from the proofs. No explicit unstable input together with its two minimizers is claimed, and no all-degree or real-only classification is supplied.

AI tools were used extensively in deriving, repairing, checking, literature auditing, and writing this work, including independent adversarial AI-agent reviews. These reviews are not human peer review. This is an unrefereed preprint, without formal proof-assistant certification; the named author is responsible for the claims and citations. No outside individual was contacted during the research. The paper and accompanying original materials are licensed under Creative Commons Attribution 4.0 International.

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