

Nonmonotone Argument and Radial Angle of a Support Slit

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Abstract

We give negative answers to the two slit-monotonicity questions of P. L. Duren recorded as Problem 6.72 in Hayman–Lingham. Every global maximizer of the real part of $a_2 + i10^{-24}(a_4 - 3a_3)$ in the normalized schlicht class omits an arc with nonmonotone continuous argument and nonmonotone radial angle, under both signed and ordinary unsigned conventions. The proof uses classical support-point geometry, a quantitative area-theorem estimate for the actual slit tip, and controlled inversion of the Schiffer differential. It establishes actual extremizers by compactness and does not assume that an arbitrary trajectory is an extremal slit. No formula for the maximizing function or uniqueness is required. This is an unrefereed AI-assisted research note.

1 The question and precise statement

Let S consist of analytic injective maps on $\mathbb{D} = \{|z| < 1\}$ normalized by

$$f(z) = z + \sum_{n \geq 2} a_n(f) z^n.$$

A support point maximizes $\operatorname{Re} L$ for a continuous complex-linear functional on $H(\mathbb{D})$ whose real part is nonconstant on S . Duren’s questions ask whether the omitted analytic arc of a support point must have monotone argument and monotone angle between its radius and tangent vectors [1, Problem 6.72]. Monotone here allows either direction. Argument means a continuous real-valued lift; tangent vectors are directed outward toward infinity.

Theorem 1. *Set*

$$\varepsilon = 10^{-24}, \quad \eta = 10^{-12}, \quad L(f) = a_2(f) + i\varepsilon(a_4(f) - 3a_3(f)).$$

Global maximizers of $\operatorname{Re} L$ on S exist. Every such maximizer omits an arc Γ on which continuous argument, signed outward radial angle, and the ordinary nonnegative outward radial angle are all nonmonotone.

We use compactness of S , the coefficient inequalities $|a_n| \leq n$ [5], and the exterior area theorem. The classical support-slit theorem gives a single analytic omitted arc from a finite tip to infinity, with increasing modulus, satisfying

$$Q(w) dw^2 > 0, \quad Q(w) = \frac{\Phi(w)}{w^2}, \quad \Phi(w) = L\left(\frac{f^2}{f-w}\right). \quad (1)$$

This is Duren’s convention [2, pp.352–355]. Pfluger’s original geometric proof is in [3, §§4.1–4.5]; his differential has the opposite sign. We use these established theorems with their stated hypotheses, and derive the needed local trajectory uniqueness below.

2 Uniform control of actual extremizers and their tips

Coefficient maps are continuous on $H(\mathbb{D})$ by Cauchy’s formula. Compactness supplies a maximizer f . The Koebe function $k(z) = z/(1-z)^2$ and the identity give real functional values 2 and 0, so every maximizer is a genuine support point. Coefficient bounds imply

$$\operatorname{Re} a_2 \geq 2 - 13\varepsilon, \quad |a_2 - 2|^2 \leq 52\varepsilon, \quad 0 \leq 1 - |a_2|^2/4 \leq 13\varepsilon. \quad (2)$$

In particular $|a_2 - 2| < 8\eta$.

The normalized odd root $g(z)^2 = f(z^2)$ is univalent. Indeed equality of two g -values implies equality of the squared inputs; oddness excludes the opposite pair except at zero. The exterior map

$$\frac{1}{g(1/\zeta)} = \zeta - \frac{a_2}{2}\zeta^{-1} + \left(\frac{3a_2^2}{8} - \frac{a_3}{2}\right)\zeta^{-3} + \dots$$

therefore satisfies the area theorem. It gives

$$|a_3 - 3a_2^2/4| \leq 2\sqrt{13/3}\eta < 5\eta, \quad |a_3 - 3| < 29\eta.$$

With

$$A = 2a_3 + a_2^2 - 6a_2, \quad B = 3a_2 - 3,$$

we obtain

$$|A + 2| < 106\eta, \quad |B - 3| < 24\eta. \quad (3)$$

The first follows from $A + 2 = 2(a_3 - 3) + (a_2 - 2)(a_2 - 4)$ and $|a_2 - 4| \leq 6$. All bounds hold uniformly for every maximizer.

We next bound the *actual* tip, without assuming continuity of tips under perturbation. Write

$$H(z) = \sqrt{z/f(z)} = 1 - \frac{a_2}{2}z + \sum_{n \geq 2} c_n z^n, \quad H_0(z) = 1 - z.$$

The same exterior map equals $\zeta H(\zeta^{-2})$, so

$$\sum_{n \geq 2} (2n - 1)|c_n|^2 \leq 13\varepsilon.$$

For $|z| \leq r = 2/3$, weighted Cauchy–Schwarz gives

$$|H - H_0| < 5\eta, \quad |H' - H'_0| < 10\eta. \quad (4)$$

The two tail sums are bounded respectively by

$$\sum_{n \geq 2} \frac{r^{2n}}{2n - 1} \leq \frac{r^4}{3(1 - r^2)} = \frac{16}{135} < \frac{1}{4}, \quad \sum_{n \geq 2} \frac{n^2 r^{2n-2}}{2n - 1} \leq (1 - r^2)^{-2} - 1 = \frac{56}{25} < \frac{9}{4}.$$

Together with $\sqrt{13} < 4$, these prove (4). At $z_* = -2/3$, $H_0 = 5/3$ and

$$\frac{3}{2} < |H| < \frac{11}{6}, \quad |H + H_0| \leq \frac{10}{3} + 5\eta < \frac{7}{2}.$$

The identities $f = z/H^2$ and $f' = (H - 2zH')/H^3$ give

$$|f(z_*) - k(z_*)| < 2\eta, \quad |f'(z_*) - k'(z_*)| < 8\eta. \quad (5)$$

Explicitly, the reciprocal-square error is at most $(14/5)\eta$, before multiplication by $|z_*| = 2/3$. The derivative numerator differs from $1/3$ by less than 19η , contributing less than 6η after division by $|H|^3$. The remaining contribution is less than 2η , since

$$|H^2 + HH_0 + H_0^2| < 11, \quad |H|^3 H_0^3 > 125/8.$$

For a conformal map $f : \mathbb{D} \rightarrow \Omega$, Schwarz's lemma applied to the inverse on any inscribed disk about $f(z)$ gives

$$\text{dist}(f(z), \partial\Omega) \leq (1 - |z|^2)|f'(z)|.$$

The direction of this inequality is essential here. Using $k(z_*) = -6/25$, $k'(z_*) = 9/125$, (5) gives

$$r_{\text{tip}} = \min_{w \in \Gamma} |w| \leq |f(z_*)| + \frac{5}{9}|f'(z_*)| < \frac{7}{25} + 7\eta < \frac{3}{10}. \quad (6)$$

The omitted arc is closed, and its increasing modulus makes its minimum the finite tip.

3 The natural coordinate and the actual omitted subarc

Coefficient extraction through degree four in $-f^2/w - f^3/w^2 - f^4/w^3 + \dots$ gives exactly

$$Q(w) = -\frac{a}{w^3} - \frac{b}{w^4} - \frac{c}{w^5}, \quad a = 1 + i\varepsilon A, \quad b = i\varepsilon B, \quad c = i\varepsilon. \quad (7)$$

For omitted w , this is (1); elsewhere it is its meromorphic continuation. In particular $a = L(f^2) \neq 0$. Under the double cover $w = -v^{-2}$,

$$Q(w) dw^2 = 4P(v) dv^2, \quad P(v) = a - bv^2 + cv^4.$$

Let $P_0(v) = -2 - 3v^2 + v^4$. From (3), on $|v| \leq 3$,

$$|P - 1 - i\varepsilon P_0| \leq 322\varepsilon\eta, \quad |P_0| \leq 110, \quad |P - 1| \leq 111\varepsilon. \quad (8)$$

Choose the analytic square root $p = \sqrt{P}$ with positive real part and put $G(v) = \int_0^v p(s) ds$. The identity $(p-1)(p+1) = P-1$ gives

$$|p-1| \leq 111\varepsilon, \quad |G(v) - v| \leq 333\varepsilon.$$

For every $|t| < 2$, Rouché on $|v| = 3$ gives exactly one root of $G(v) = t$. Since $G' = p \neq 0$, the roots patch to a holomorphic inverse V , with

$$V(0) = 0, \quad V'(0) = a^{-1/2}, \quad |V(t) - t| \leq 333\varepsilon, \quad V'(t) = 1/p(V(t)). \quad (9)$$

Both G and V are odd.

For $0 < t \leq T = 7/4$, define $W(t) = -V(t)^{-2}$. It is regular,

$$Q(W)W'^2 = 4, \quad W(t) \sim -a/t^2, \quad |W(t)| > 25/81 > 3/10, \quad (10)$$

because $|V(t)| < 9/5$. It has no self-intersection: equality of two W -values implies $V(t_1) = \pm V(t_2)$, and the odd inverse relation forces $t_1 = \pm t_2$.

We now identify W with the actual omitted slit. A distant tail of Γ lifts to $v \rightarrow 0$ in the double cover. The positive trajectory condition says $\Im G$ is constant; its limiting value is zero. The two real G -rays project to the same geometric w -arc by oddness of V . Thus W and Γ coincide near infinity. The set

$$E = \{t \in (0, T] : W(t) \in \Gamma\}$$

is nonempty and relatively closed. At every member, (6) and (10) exclude the actual tip, with radius gap $7/810$. Moreover Q is finite and nonzero there. A local square-root primitive of Q makes positive trajectories horizontal lines, so local uniqueness makes E relatively open. Connectedness gives $E = (0, T]$. No converse extremality criterion for arbitrary trajectories has been used.

Put $Z = tV'/V$. Independently of the fine expansion below,

$$Z(t) = \frac{G(V(t))/V(t)}{p(V(t))}, \quad |Z(t) - 1| \leq \frac{222\varepsilon}{1 - 111\varepsilon} < 223\varepsilon \quad (0 < t \leq T). \quad (11)$$

Here $G(v)/v$ is the average of p on a radial segment. Thus $\operatorname{Re} Z > 0$, and $d \log |W|/dt = -2 \operatorname{Re} Z/t < 0$: increasing t is inward, so $-W'$ is the outward tangent.

4 Certified reversal of both angles

Define

$$d(t) = t + \frac{1}{2}t^3 - \frac{1}{10}t^5, \quad q(t) = d'(t) - d(t)/t = t^2 - \frac{2}{5}t^4.$$

We claim, uniformly for $1 \leq t \leq T$,

$$|V - t - i\varepsilon d(t)| \leq 487\varepsilon\eta, \quad |V' - 1 - i\varepsilon d'(t)| \leq 162\varepsilon\eta. \quad (12)$$

Here are explicit analytic remainder bounds. On the selected square-root branch,

$$\sqrt{1+x} - 1 - x/2 = -\frac{x^2}{2(\sqrt{1+x} + 1)^2}.$$

Together with (8), it gives

$$|p - 1 - (i\varepsilon/2)P_0| \leq 162\varepsilon\eta.$$

Indeed the costs are $161\varepsilon\eta + (111^2/2)\varepsilon^2$. Since $P_0 = -2d'$, integration gives $G(v) = v - i\varepsilon d(v) + R(v)$ with $|R| \leq 486\varepsilon\eta$. The bound $|d'| \leq 55$ on $|v| \leq 3$ and (9) make inversion cost at most $18315\varepsilon^2$, proving the first inequality in (12).

Complex Taylor's integral remainder gives

$$|(1+x)^{-1/2} - 1 + x/2| \leq 2|x|^2 \quad (|x| \leq 1/4).$$

For example its second derivative has modulus at most $(3/4)(4/3)^{5/2} < 4$. The coefficient uncertainty costs $161\varepsilon\eta$; the quadratic term costs $24642\varepsilon^2$. Since $|P'_0| \leq 126$, replacing $P_0(V)$ by $P_0(t)$ costs $20979\varepsilon^2$. The total is at most $162\varepsilon\eta$, proving the second inequality. All absorptions are justified by

$$(111^2/2)\eta < 1, \quad 18315\eta < 1, \quad 45621\eta < 1.$$

Writing the two remainders in (12) as $R_V, R_{V'}$, subtraction before division gives the exact identity

$$V(Z - 1 - i\varepsilon q) = tR_{V'} - R_V - i\varepsilon q(V - t).$$

On this interval $|V| > 1/2$ and $|q| < 7$. Hence

$$|Z - 1 - i\varepsilon q| \leq 1541\varepsilon\eta + 4662\varepsilon^2 < 2000\varepsilon\eta. \quad (13)$$

The exact values

$$q(1) = 3/5, \quad q(T) = -441/640, \quad 2000\eta = 2 \cdot 10^{-9}$$

therefore imply

$$\operatorname{Im} Z(1) > \varepsilon/2, \quad \operatorname{Im} Z(T) < -\varepsilon/2.$$

For a continuous argument θ of W ,

$$\theta'(t) = \operatorname{Im}(W'/W) = -2 \operatorname{Im} Z(t)/t.$$

Its derivative is negative at 1 and positive at T , both regular points of the actual slit. It cannot be monotone. The signed outward radial angle is

$$\beta(t) = \arg((-W')/W) = \arg Z(t).$$

The branch is fixed by (11); it tends to zero as $t \downarrow 0$, since $V(t) = a^{-1/2}t + O(t^3)$. It is positive at 1 and negative at T . A nondecreasing function with initial limit zero cannot take the negative value, and a nonincreasing one cannot take the positive value. Thus the signed angle is nonmonotone. Continuity gives a zero between these two points, whereas $|\beta|$ is positive at both endpoints. The ordinary unsigned angle is therefore nonmonotone even on that compact subarc. An inward tangent gives its supplement, and the acute tangent-line interpretation gives $|\beta|$; neither restores monotonicity. This proves Theorem 1.

5 Attribution, scope and verification

The support-slit geometry and Schiffer method are classical, as are coefficient-extremal and Loewner–Schiffer control methods [4]. Duren and Brown established positive monotonicity results for particular evaluation and lower-degree coefficient functionals [2, 6, 7]. Zemyan's tangent-waver construction [8] uses Brown's actual point-evaluation extremizers, whose polar argument and radial angle remain monotone. Tangent direction $\arg w'$ is a different observable from $\arg w$ and $\arg w' - \arg w$. The complete later papers [9, 10] investigate finiteness and bounds for hypothetical polar reversals; their examples do not supply an actual support point with either target failure. Our proof does not use their count bounds.

Duren–Leung–Schiffer [11] establish strict tip-angle inequality for finite coefficient functionals. The classification studied in Zhang–Ma [12] concerns saturation of the tip-angle bound; it does not imply monotonicity in the interior. Rationality of a functional and rationality of its maximizer must also be distinguished [13, 14]. These prior results do not give the fourth-degree reversal mechanism proved here. Goh's coefficient two-functional theorems [15] require a second independent supporting or Schiffer-critical functional; maximization of our single L does not supply that hypothesis. The complete paper

contains no verified example answering either monotonicity question negatively. The present argument controls every maximizer without an extremal solver, a unique maximizing selection, or boundary-tip continuity. At $\varepsilon = 0$ the sign reversal disappears.

The bounded audits compare complete primary constructions and proofs. No earlier negative answer to the exact target questions was verified in the inspected corpus. The later Duren commentary [16] and Hirschweiler's 1995 support-geometry paper [17] could not be inspected. Priority therefore cannot be fully cleared. Other uninspected leads and the precise limits of the comparisons are recorded in the companion assessments. We make no claim of historical firstness or of the questions' continued open status in 2026.

The supporting archive contains the manuscript source, immutable inputs, independent analytic and quantitative derivations, rational and interval controls, source-comparison receipts, and integrity seals. Third-party full texts are excluded. Finite controls supplement the uniform proofs; they are not formal certification of the classical inputs. AI tools were used extensively in development, writing, reproduction and adversarial verification. This note is unrefereed and has received no human peer review. Internal reviews are AI-assisted.

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