

A torus-collar proof of the real-spectrum compact-support classification for Nijenhuis operators

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Abstract

We give a complete smooth compact-support proof of the classification asked for in the historical integrable-systems problem source: a real Jordan type occurs on a nonempty open subset of $\mathbb{R}^n$ for a compactly supported smooth Nijenhuis operator exactly when its spectrum is real. The construction realizes arbitrary specified numerical real Jordan data in a commuting local frame. On a same-dimensional torus collar, the torsion equation reduces to $(A - \lambda I)A' = 0$; an ordered finite path removes every Jordan block and reaches an exact zero band. The nonreal obstruction and earlier semisimple and two-dimensional examples are prior results. We include the precise zero-point rigidity deduction needed for necessity. The historical priority of the full result and of this construction remains unestablished. Extensive AI tools assisted solving, verification and writing. This preprint is unrefereed and has no external human peer review or formal proof-assistant certification.

1 Statement, historical target and attribution

For a smooth real field L of tangent-space endomorphisms, use the convention

$$N_L(X, Y) = L^2[X, Y] + [LX, LY] - L[LX, Y] - L[X, LY].$$

An operator is Nijenhuis when $N_L = 0$. Jordan type here means the block sizes and grouping by eigenvalue; our existence statement also fixes any desired numerical real eigenvalues.

Theorem 1. *For every $n \geq 1$ and every prescribed real Jordan matrix J with entirely real spectrum, there exists a globally C^∞ , compactly supported real Nijenhuis operator on $\mathbb{R}^n$ whose matrix is J in a smooth commuting local frame on a nonempty open domain. Conversely, every eigenvalue at every point of a compactly supported smooth real Nijenhuis operator on $\mathbb{R}^n$ is real.*

The type, rank and eigenvalues may change outside the realization domain. The theorem prescribes neither the matrix in the ambient Cartesian frame nor global regularity, and is a smooth, not a real-analytic, statement. The zero matrix is permitted.

The target is Problem 5.11 on p. 29 of the 14 January 2020 author-v2 of Bolsinov–Matveev–Miranda–Tabachnikov [1]; its 10 April 2018 author-v1 counterpart is Problem 4.10 on p. 37. The original journal work appeared in 2018. A corpus label “5.9” is not the locator in either inspected version. The full wording asks for necessary and sufficient conditions for compact support with a prescribed unchanged algebraic type on a domain. Theorem 1 answers that printed question, without asserting that it remained open worldwide before this note.

The source already credits the nonreal obstruction, real semisimple realizations and a two-dimensional Jordan example. Bolsinov–Konyaev–Matveev (BKM) [2], author-v2 Theorem 4.5 and the following p. 28 remark, also give arbitrary algebraic types at local points. Those facts are prior. The proof below supplies an explicit fixed-type open domain together with a smooth compact-support transition. The bounded literature audit in the supplement certifies neither firstness of the full theorem nor priority of the torus-collar mechanism.

2 The one-variable collar equation

Let $m = n - 1 \geq 1$ and work on $\mathbb{T}^m \times (-1, 1)$. Write $e_1, \dots, e_m$ for the invariant angular fields and $v = \partial_t$. They form a global commuting frame. Angular coordinate functions are only locally real-valued; the invariant fields and their dual forms are global. For smooth matrix, vector and scalar functions of t , set

$$Le_i = \sum_{k=1}^m A_{ki}(t)e_k, \quad Lv = \sum_{k=1}^m b_k(t)e_k + \lambda(t)v.$$

All angular-pair torsions vanish. For a mixed pair the brackets are

$$[Le_i, Lv] = -\lambda A' e_i, \quad [Le_i, v] = -A' e_i, \quad [e_i, Lv] = [e_i, v] = 0.$$

Consequently

$$N_L(e_i, v) = (A - \lambda I)A' e_i, \quad N_L = 0 \iff (A - \lambda I)A' = 0. \quad (1)$$

Tensoriality and skew symmetry account for every component. The product order is essential. There is no condition on b' , λ' , rank, diagonalizability or invertibility.

Lemma 1. *For every real-spectrum Jordan matrix J of size $n \geq 2$, after a constant permutation of whole blocks there is a smooth path $L(s) = \begin{pmatrix} A(s) & b(s) \\ 0 & \lambda(s) \end{pmatrix}$, $0 \leq s \leq 1$, from J to zero, constant near both endpoints, satisfying $(A - \lambda I) dA/ds = 0$.*

Proof. Use upper-superdiagonal Jordan convention and designate the last vector of one block as transverse. The remaining hyperplane is invariant. Thus $J = \begin{pmatrix} A_0 & b_0 \\ 0 & \lambda_0 \end{pmatrix}$, where A_0 contains the other blocks and the distinguished block shortened by one; a size-one distinguished block disappears. The vector b_0 retains the removed terminal arrow, so the initial full matrix is exactly J .

First hold A and λ fixed and move b_0 to zero. Whenever A is fixed, both b and λ can move freely in (1). Process the blocks of A_0 one at a time, leaving processed blocks zero and unprocessed blocks fixed. For a current block $\mu I_r + K$, move λ to the real number μ while holding A fixed. Write

$$K = \sum_{j=1}^{r-1} a_j E_{j,j+1}, \quad KK' = \sum_{j=1}^{r-2} a_j a'_{j+1} E_{j,j+2}.$$

Initially every $a_j = 1$. With $\lambda = \mu$, move a_1 to zero, then a_2 to zero, and so forth. Only one arrow changes at a time. Whenever $a'_{j+1} \neq 0$, its predecessor a_j is already identically zero; the first arrow has no predecessor. Hence $KK' = 0$. Other blocks have zero derivative and contribute nothing to the ordered product.

The active block is now scalar. Move its scalar value $c(s)$ from μ to zero and set $\lambda(s) = c(s)$ on this stage. The derivative is supported in that block, where $A - \lambda I$ vanishes. Repeat for every angular block. For singleton blocks there are no arrows; zero eigenvalues allow constant scalar stages. At the end $A = b = \lambda = 0$.

There are finitely many stages. On separate successive intervals use a C^∞ switch constant near both endpoints. For example, with $H(u) = e^{-1/u}$ for $u > 0$ and $H(u) = 0$ otherwise, the function $S(u) = H(u)/(H(u) + H(1 - u))$ is smooth, zero for $u \leq 0$ and one for $u \geq 1$; apply S to an affine parameter whose endpoints extend beyond $[0, 1]$. Each stage then has genuine open endpoint plateaus. Values agree at joins, so concatenation is smooth. Reparametrization only multiplies A' by a scalar, preserving (1). No division by an eigenvalue or eigenvalue difference is used. $\square$

Choose a smooth even $q : (-1, 1) \rightarrow [0, 1]$ equal to zero for $|t| \leq 1/4$ and one for $|t| \geq 3/4$; explicitly take $q(t) = S(2(t^2 - 1/16))$. Substitute $s = q(t)$ in Lemma 1. The chain rule gives

$$(A(q) - \lambda(q)I) \frac{d}{dt} A(q) = q'(t)(A(q) - \lambda(q)I) \frac{d}{ds} A(q) = 0.$$

The tensor is J on the central band and exactly zero on both end bands. There is no absolute-value corner and no arbitrary scalar cutoff of J . A constant frame permutation restores any initially specified block order; constant linear combinations of commuting fields still commute.

3 A same-dimensional torus collar and extension

For every $d \geq 1$, the torus $\mathbb{T}^d$ embeds as a two-sided smooth hypersurface in $\mathbb{R}^{d+1}$. Here is the needed all-dimensional induction. Start with a circle in $\mathbb{R}^2$. If $f : \mathbb{T}^{d-1} \hookrightarrow \mathbb{R}^d$ is an oriented embedded hypersurface, it has a global unit normal ν . Inside $\mathbb{R}^{d+1}$ its rank-two normal bundle has the explicit frame $(\nu, 0), (0, 1)$. By compactness and the tubular-neighborhood theorem, a sufficiently small normal disk bundle is embedded. Its boundary is $\mathbb{T}^{d-1} \times S^1 = \mathbb{T}^d$, with parametrization

$$F(p, \phi) = (f(p) + \varepsilon \cos \phi \nu(p), \varepsilon \sin \phi).$$

The tubular coordinates recover $p, \cos \phi, \sin \phi$, proving injection and immersion. Compactness makes this an embedding. A global unit normal is $(\cos \phi \nu(p), \sin \phi)$: orthogonality follows from $\nu \cdot df = \nu \cdot d\nu = 0$. This verifies two-sidedness for the next step and completes the induction.

For $n \geq 2$ a thin normal collar of the embedded $\mathbb{T}^{n-1} \subset \mathbb{R}^n$ is diffeomorphic, after rescaling, to $\mathbb{T}^{n-1} \times (-1, 1)$. Push the preceding tensor through this diffeomorphism. Its frame still commutes, and torsion vanishes by naturality. Its support lies in the image of the compact set $\mathbb{T}^{n-1} \times [-3/4, 3/4]$, strictly inside the open collar. Extend it by zero to $\mathbb{R}^n$. Each collar-boundary or exterior point has a neighborhood outside that compact support. Thus the extension is globally smooth, and its torsion is zero by locality. No exterior angular frame is required. The central band is nonempty and open; a connected coordinate box inside it is a realization domain.

For $n = 1$, a compactly supported smooth scalar function equal to the desired eigenvalue on an interval, multiplied by Id , suffices. The alternating vector-valued two-form N_L vanishes in dimension one. The zero matrix is also realized by the zero tensor. This proves the existence direction in every stated boundary case.

4 The credited nonreal obstruction

BKM's author-v2 Theorem 6.1 [2], pp. 41–43, states nonreal spectral persistence with constant algebraic multiplicity on a closed connected smooth manifold. It has no analyticity or regularity hypothesis. The following weaker consequence is all that is needed here. We give its maximum-principle justification using their local splitting and canonical complex/holomorphic results (Theorems 3.1 and 3.3).

Lemma 2. *A smooth real Nijenhuis operator on a closed connected manifold which vanishes at one point has entirely real spectrum everywhere.*

Proof. Suppose otherwise. Let $\mu(x)$ be the largest positive imaginary part of an eigenvalue, with value zero for an all-real spectrum. Roots depend continuously on polynomial coefficients, so μ is continuous. On the compact manifold it has a positive maximum μ_0 . Among roots on the line $\text{Im } z = \mu_0$ at maximum points choose the largest algebraic multiplicity m_0 . The locus E having such a root of multiplicity m_0 is nonempty and closed: along a convergent sequence choose a convergent subsequence of those roots; vanishing polynomial derivatives shows multiplicity at least m_0 , and maximality gives equality.

At $p \in E$, isolate the root λ_0 and its conjugate by the local splitting theorem. On this real block the canonical complex structure makes its complex characteristic factor $P(z) = z^{m_0} + a_1 z^{m_0-1} + \dots + a_{m_0}$ holomorphic in the block's complex coordinates and independent of complementary coordinates. At p , $P(z) = (z - \lambda_0)^{m_0}$. All nearby roots have imaginary part at most μ_0 . Therefore $-\text{Im } a_1 \leq m_0 \mu_0$ with equality at p . The holomorphic maximum principle makes a_1 constant locally. Every root consequently has imaginary part μ_0 . The shifted polynomial $P(w + i\mu_0)$ has only real roots. Its coefficients are real-valued holomorphic functions and hence constant. Thus P remains $(z - \lambda_0)^{m_0}$ locally. The complementary block can be empty; otherwise independence of its coordinates gives the same conclusion on the product neighborhood. Hence E is open. Connectedness gives $E = M$, contradicting the zero point. $\square$

For compactly supported L on $\mathbb{R}^n$, identify $\mathbb{R}^n$ with a standard chart of S^n and extend by zero near infinity. Compact support makes this a smooth Nijenhuis field on closed connected S^n , zero on a neighborhood of infinity. Lemma 2 excludes nonreal spectrum, completing Theorem 1.

The attribution is substantive: both the obstruction and its local splitting/holomorphic mechanism are prior. Our argument avoids relying on the final polynomial-reduction sentence of the author-v2 proof, whose literal count of removed nonreal pairs is imprecise. For example, rotation blocks with spectra $\{\pm i, \pm i/2\}$ all acquire real eigenvalues under $z \mapsto z^2 + 1$, although two original pairs were nonreal. This is not a counterexample to the theorem; it explains why the exact application is proved by the preceding maximum argument. We do not certify every step of the stronger persistence proof or identify the unread *Advances in Mathematics* publisher body with author-v2 bytes.

5 Relation to prior constructions and verification

The arbitrary-type point construction of BKM Theorem 4.5 and its remark is generically semisimple in independent characteristic-coefficient coordinates. Repeated-root conditions there do not supply a fixed nonsemisimple type on an open domain. In *Geometry III* [3], the actual journal regular-layer theorem is Theorem 1.10 (author-v1 Theorem 1.4). It realizes regular layers near a nonzero nilpotent block, rather than arbitrary repeated blocks with a compact zero transition. The journal and arXiv numberings must be distinguished.

Geometry IV [4] does allow singular or zero output symmetries of a regular generator. Its functional coefficients nevertheless obey compatibility equations. In its size-three nilpotent formula an unwrapped angular coordinate multiplies a transverse derivative, so naive periodization does not provide the collar construction. This observation does not exclude a different use of that framework. Local constant-type or spectral splitting results and fixed functional calculus likewise do not by themselves supply compact support: a product block sum retains unbounded cylinders

of support, and fixed calculus preserves diagonalizability of a semisimple input. Standard tubular-neighborhood and smooth-switch arguments are not claimed as new.

The accompanying portable standard-library controls check the full collar identity from coordinate first jets, finite ordered Jordan schedules with exact polynomial arithmetic, and deliberately invalid arrow order, cutoff and unmatched-scalar variants. They corroborate indexing; the all-dimensional proof is above. These are newly distributed controls, not a reconstruction or certification of the submitted historical omitted programs or their counts. The source ledger records actual prior reading scopes and unknowns, including unread 2025 Sbornik and 2026 cohomology bodies and older cited works. No exhaustive historical inference is made.

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