

A negative-curvature normal barrier on an algebraic cone that is not a hyperbolicity cone

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Abstract

We give an explicit affirmative answer to Tunçel’s 2017 existential question about algebraic convex cones with negative-curvature normal barriers. The one-sided power cone $K = \{(x, y, z) \geq 0 : xy^2 \geq z^3\}$ admits the barrier $F = -25 \log(z(xy^2 - z^3))$ with parameter 100 and $D^3F(a)[h, u, u] \leq 0$ for every $a \in \text{int } K$, $h \in K$, and $u \in \mathbb{R}^3$. Exact polynomial and positive-semidefinite certificates prove the curvature inequality, including its degenerate directions. A boundary divisibility argument rules out every homogeneous hyperbolic polynomial defining K in $\mathbb{R}^3$. The cone itself is a known member of Hildebrand’s classification; the contribution here is the explicit barrier certificate and its verification for the stated existential problem.

1 The problem and the example

Tunçel [1, p. 799, Open Problem 1] asks whether an algebraic convex cone can admit a normal barrier with negative curvature without being a hyperbolicity cone. Here algebraic means defined by homogeneous polynomial inequalities. For a regular cone (closed, pointed, convex, with nonempty interior), negative curvature means

$$D^3F(a)[h, u, u] \leq 0 \quad (a \in \text{int } K, h \in K, u \in \mathbb{R}^3). \quad (1)$$

This is the third-derivative convention of [1, 2], equivalently the concavity of $a \mapsto DF(a)[h]$ for fixed $h \in K$ [3, Definition 3.1]. A normal barrier is a logarithmically homogeneous self-concordant barrier; we also verify its gradient bound directly.

Theorem 1. *Let*

$$K = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0, p(x, y, z) := xy^2 - z^3 \geq 0\}.$$

Then K is a regular algebraic convex cone. On $\text{int } K$, the function

$$f = -\log p - \log z, \quad F = 25f$$

is a normal barrier with parameter 100 satisfying (1). No homogeneous polynomial in three variables has K as its hyperbolicity cone.

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The underlying cone appears in Hildebrand [4, Theorem 3.2, family 5]: take the conjugate exponents $3, 3/2$ and coordinates $(x_1, x_2, x_3) = (x, z, y)$. Self-concordant barriers for this same nonnegative power-cone family are also known: Roy–Xiao [5, Theorem 2] give, for this cone, the parameter-3 barrier

$$B = -\log(x^{1/3}y^{2/3} - z) - \frac{2}{3}\log x - \frac{1}{3}\log y - \log z.$$

Our parameter 100 is not an improvement of their self-concordance parameter. The property established here is the negative-curvature certificate for the displayed polynomial-logarithmic barrier, together with the all-polynomial hyperbolicity obstruction. The theorem concerns this one-sided cone and the original existential question. It does not resolve the broader question whether every regular cone admits such a barrier [3, Section 6]. All arguments below are analytic; the accompanying exact computations are independently checkable supplements.

2 Geometry and normality

The displayed homogeneous inequalities make K closed and conic. The nonnegative orthant makes it pointed; $(2, 1, 1)$ is strictly feasible, and

$$\text{int } K = \{x, y, z > 0 : xy^2 > z^3\}.$$

For $m(x, y) = x^{1/3}y^{2/3}$ on the positive orthant,

$$\nabla^2 m = -\frac{2}{9}x^{-5/3}y^{-4/3} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \preceq 0.$$

Concavity extends to the closed orthant by continuity. Thus $K = \{x, y \geq 0, 0 \leq z \leq m(x, y)\}$ is convex.

Put $g = z^3/y^2$ and $\delta = x - g > 0$. Then $f = -\log \delta - 2\log y - \log z$. At a fixed base point, for any direction u , define the linear functionals

$$b_u = u_y/y, \quad c_u = u_z/z, \quad w_u = \frac{u_x - g(3c_u - 2b_u)}{\delta}, \quad v_u = \sqrt{6g/\delta}(c_u - b_u), \quad \ell_u = c_u - 4b_u.$$

Direct differentiation gives

$$\|u\|_f^2 := D^2 f[u, u] = w_u^2 + v_u^2 + 2b_u^2 + c_u^2, \tag{2}$$

$$D^3 f[u, u, u] = -2w_u^3 - 3w_u v_u^2 + \ell_u v_u^2 - 4b_u^3 - 2c_u^3. \tag{3}$$

For example, $Dg[u] = g(3c_u - 2b_u)$, $D^2 g[u, u] = 6g(c_u - b_u)^2$, and $D^3 g[u, u, u] = 6g(c_u - b_u)^2(c_u - 4b_u)$, which yield these identities. Equation (2) is positive definite: zero forces $b_u = c_u = w_u = 0$ and hence $u = 0$.

For completeness we prove the mixed form of self-concordance used in [3, Section 2.2]. Polarizing (3) gives

$$\begin{aligned} D^3 f[h, u, u] = & -2w_h w_u^2 - w_h v_u^2 - 2w_u v_h v_u \\ & + \frac{1}{3}(\ell_h v_u^2 + 2\ell_u v_h v_u) - 4b_h b_u^2 - 2c_h c_u^2. \end{aligned}$$

Weighted Cauchy–Schwarz gives $|\ell_a| \leq 3\sqrt{2b_a^2 + c_a^2} \leq 3\|a\|_f$. The four groups in this display have absolute values bounded respectively by 2, 3, 3, 2 times $\|h\|_f \|u\|_f^2$. For the last bound, use

$$|4b_h b_u^2 + 2c_h c_u^2| \leq 2\sqrt{2b_h^2 + c_h^2} \sqrt{2b_u^4 + c_u^4} \leq 2\|h\|_f \|u\|_f^2.$$

Consequently $|D^3 f[h, u, u]| \leq 10 \|h\|_f \|u\|_f^2$. Since $\|u\|_F = 5 \|u\|_f$, scaling by 25 gives

$$|D^3 F[h, u, u]| \leq 2 \|h\|_F \|u\|_F^2,$$

including the standard scalar self-concordance inequality when $h = u$.

The polynomial zp has degree four, so $F(ta) = F(a) - 100 \log t$. Differentiation yields $\nabla^2 F(a)a = -\nabla F(a)$ and $a^T \nabla^2 F(a)a = 100$; hence

$$\nabla F(a)^T \nabla^2 F(a)^{-1} \nabla F(a) = 100.$$

At every finite boundary point of K , either $z = 0$ or $p = 0$. Thus along every interior sequence tending to that point, $zp \rightarrow 0^+$ and $F \rightarrow +\infty$. Extended by $+\infty$ outside $\text{int } K$, F is a closed convex function. These facts prove its claimed normality.

3 The negative-curvature certificate

The diagonal maps $D = \text{diag}(\alpha, \beta, \gamma)$ with positive entries and $\alpha\beta^2 = \gamma^3$ are automorphisms of K , and $f(Dq) = f(q) - 4 \log \gamma$. For a fixed interior point (x, y, z) , take $D = \text{diag}(y^2/z^3, 1/y, 1/z)$. It sends that point to $(r, 1, 1)$, where $r = xy^2/z^3 > 1$. Differentiating with this D held fixed gives

$$D^3 f(q)[h, u, u] = D^3 f(Dq)[Dh, Du, Du].$$

It is therefore enough to prove the matrix inequality at $(r, 1, 1)$ for every $r > 1$.

Every $h \in K$ with $h_y > 0$ has the decomposition

$$h = h_y(t^3, 1, t) + (h_x - h_z^3/h_y^2)e_x, \quad t = h_z/h_y \geq 0,$$

with both coefficients nonnegative. If $h_y = 0$, then $h_z = 0$ and h is a nonnegative multiple of e_x . By linearity in h , it remains to check $h(t) = (t^3, 1, t)$ for $t \geq 0$, and e_x .

Set $s = r - 1 > 0$ and $M = -s^3 D^3 f(r, 1, 1)[h(t)]$. Its symmetric entries are

$$\begin{aligned} M_{11} &= 2(t-1)^2(t+2), \\ M_{12} &= 2(t-1)(2t^2+2t-1-3r), \\ M_{13} &= 6(t-1)(r-t^2-t+1), \\ M_{22} &= 4r^3+12r^2-18r^2t+6rt^3-6rt+2t^3, \\ M_{23} &= 12r^2t-18r^2-6rt^3+24rt-6r-6t^3, \\ M_{33} &= 6(r+2)t^3+2(r^3-6r^2-18r-4)t+12r(r+2). \end{aligned}$$

These follow, for instance, by differentiating $f_{ij} = p_i p_j / p^2 - p_{ij} / p + \mathbf{1}_{i=j=z} / z^2$. The next two leading minors are the following polynomial identities:

$$M_{11}M_{22} - M_{12}^2 = 4s(t-1)^2 P, \tag{4}$$

$$\det M = 8s^4(t-1)^2 R, \tag{5}$$

where

$$\begin{aligned} P &= 2(t+2)s^2 - 3(t-1)(3t+5)s + 3(t-1)^2(t+2)^2, \\ R &= (t+2)(2ts^2 - 3(t-1)(3t+1)s + 3(t-1)^2(t^2+4t+1)). \end{aligned}$$

For $0 \leq t < 1$, every term in these expressions is nonnegative and the constant terms are positive. At $t = 1$, $P = R = 6s^2 > 0$. For $t \geq 1$ use the exact square identities

$$8(t+2)P = [4(t+2)s - 3(t-1)(3t+5)]^2 + 3(t-1)^2(8t^3 + 21t^2 + 6t - 11), \quad (6)$$

$$\frac{8tR}{t+2} = [4ts - 3(t-1)(3t+1)]^2 + 3(t-1)^3(8t^2 + 13t + 3). \quad (7)$$

The first remainder polynomial, at $t = 1 + v$, equals $8v^3 + 45v^2 + 72v + 24 > 0$ for $v \geq 0$. Both remainders are strictly positive for $t > 1$, and all divisors used are positive. Hence $P, R > 0$ throughout $s > 0, t \geq 0$. If $t \neq 1$, the first three leading minors are positive, so Sylvester's criterion gives $M \succ 0$.

At the exceptional direction $t = 1$, substitution instead gives

$$M = 2s^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2r+1 & -3 \\ 0 & -3 & r+2 \end{pmatrix} \succeq 0,$$

because the lower block has positive diagonal and determinant $(2r+1)(r+2) - 9 = (r-1)(2r+7) > 0$. For the remaining endpoint there is a direct Gram certificate:

$$-s^3 D^3 f(r, 1, 1)[e_x] = 2aa^T + 6sbb^T \succeq 0, \quad a = (1, 2, -3)^T, \quad b = (0, 1, -1)^T.$$

Thus all cone directions are covered. Congruence under D , linearity in h , and multiplication by 25 prove (1).

4 Excluding every hyperbolicity polynomial

Recall that a homogeneous polynomial H is hyperbolic in e if $H(e) > 0$ and $H(v + te)$ has only real roots for every v . Its closed hyperbolicity cone is the closure of the positive-eigenvalue region in the convention of [1]. The direction e lies in the interior of that cone. Boundary eigenvalues are nonnegative by continuity from the positive region. The boundary lies in $\{H = 0\}$: if all eigenvalues there were positive, continuity of roots (with fixed nonzero leading coefficient) would give an interior neighborhood.

Suppose that this cone were K . Then $e \in \text{int } K$, and the curved boundary patch implies

$$H(z^3/y^2, y, z) = 0 \quad (y, z > 0).$$

Clearing denominators gives a polynomial identity on an open set in $\mathbb{R}^2$, hence an identity everywhere. In $\mathbb{R}(y, z)[x]$ this says $x - z^3/y^2$ divides H . The polynomial $p = y^2x - z^3$ is primitive over $\mathbb{R}[y, z]$, since $\gcd(y^2, z^3) = 1$, and is irreducible over the fraction field, being linear in x . Gauss's lemma gives $p \mid H$ in $\mathbb{R}[x, y, z]$.

Write $a = (e_x e_y^2)^{1/3}$ and $b = e_z$, so $a > b > 0$. For $v = (0, 0, 1)$,

$$\begin{aligned} p(v + te) &= a^3 t^3 - (1 + bt)^3 \\ &= ((a - b)t - 1)((a^2 + ab + b^2)t^2 + (a + 2b)t + 1). \end{aligned}$$

The quadratic discriminant is $(a + 2b)^2 - 4(a^2 + ab + b^2) = -3a^2 < 0$. Therefore $p(v + te)$ has nonreal roots. Since $p \mid H$ and $H(e) > 0$, the restricted product is nonzero and these roots cannot cancel. This contradicts hyperbolicity in e , proving the final assertion of Theorem 1. The argument treats arbitrary homogeneous H , without an assumed degree or a particular defining polynomial.

Verification, attribution, and research status

The supplement supplies independently authored exact rational-polynomial checkers for the derivative, minor, square, and endpoint identities, with additional logarithm-series checks and deliberately false controls. The proof above establishes the universal inequalities; finite samples are supplementary evidence. The term $-\log z$ is substantive: for $f_0 = -\log p$, the matrix $-s^3 D^3 f_0[h]$ has determinant -24576 at base $(5, 1, 1)$ and direction $h = (8, 1, 2)$.

The cone’s prior occurrence in [4] is explicitly credited; its hyperbolic affine spheres are distinct from polynomial hyperbolicity. Definitions and related barrier results are credited to [2, 3]. A bounded primary-source priority audit accompanies the supplement; no assertion of exhaustive literature coverage or first-ever priority is made.

AI and review disclosure. AI tools were used extensively in developing, checking, and writing this result and its verification materials. This preprint is unrefereed. The accompanying independent adversarial AI reviews do not constitute external human peer review or proof-assistant certification.

References

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