Near-one oriented-percolation flow from Bernoulli first-passage estimates

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Abstract

For the square-lattice oriented Bernoulli edge-flow density, we derive $1 - v(p) \sim \sqrt{2(1-p)}$ as $p \uparrow 1$, the asymptotic posed by Aldous and Lenderman. The argument is an explicit corollary of Basdevant–Gouéré–Théret's 2021 strict-chain scaling and weighted-list estimate. The directed cut dual has zero-cost backward steps, so their ordinary first-passage theorem cannot be substituted directly. Grouping the two outgoing rare-edge marks into independent site marks gives the required event inclusion into their already evaluated counting sum. We supply this comparison, the finite directed-cut reduction and the order of limits. The asymptotic is presented as an attributed application of older estimates; no claim of novelty, historical firstness or present-day openness is made.

1 The question and the precise model

Aldous's problem page [1] describes the conjecture, formulated with Jason Lenderman in 2004, and the difficulty of justifying its Poisson–Hammersley heuristic. We consider exactly that edge-flow model. The vertices are $\{0, \dots, n-1\}^2$, $n \geq 2$, with edges pointing north or east. Independently, each edge has capacity one with probability p and capacity zero otherwise. The source vertices are the left and bottom boundary vertices except $(0, n-1)$ and $(n-1, 0)$; the sinks are the top and right boundaries. Let V_n be the maximum flow between these sets. Integrality identifies it with the maximum number of edge-disjoint open directed paths; shared vertices are permitted.

The source states the deterministic L^1 limit $V_n/(2n) \longrightarrow v(p)$. We use this stated existence result with its normalization. In fact the expectation limit used below also follows directly from diagonal superadditivity after the finite reduction. No heuristic identity from the problem page is an input.

Theorem 1. *In this model,*

$$\lim_{q \downarrow 0} \frac{1 - v(1-q)}{\sqrt{2q}} = 1.$$

Here the finite-volume limit defining v is taken at each fixed $q > 0$ before q tends to zero.

The substantial older input is the numerical weighted-list bound inside the proof of Basdevant–Gouéré–Théret (BGT) [2, §3.3, equation (33)], together with their strict-site score scaling. All equation numbers for that input refer to their public 2021 arXiv v1; publication metadata identifies a 2022 journal version. We do not assert a line-by-line identity of the two editions.

2 The finite directed-cut reduction

Put $N = n - 2$ and

$$e_L = ((0, n-2), (0, n-1)), \qquad e_B = ((n-2, 0), (n-1, 0)).$$

Write c_L, c_B for their capacities. Reflect the vertical coordinate of the bounded-face dual, obtaining $Q_N = \{0, \dots, N\}^2$. A dual east or north traversal costs the capacity of the primal edge it crosses; a reverse traversal costs zero. For example, a dual east edge starting at (x, y) crosses $((x+1, N-y), (x+1, N+1-y))$, and a dual north edge crosses $((x, N-y), (x+1, N-y))$. The correspondence is injective on unoriented dual edges. Let T_N be the minimum dual cost from $(0, 0)$ to (N, N), with $T_0 = 0$.

Lemma 1. *For every nonnegative primal capacity assignment,*

$$V_n = c_L + c_B + T_N. \tag{1}$$

Proof. Consider any cut whose source side contains all prescribed sources and no sink. Draw the crossing of every primal edge with differently labelled endpoints, orienting the interface with the source side to its right in the physical embedding. Include exterior endpoints at e_L, e_B. At each bounded face the numbers of incoming and outgoing crossings agree: the two types of binary-label transitions alternate around its four corners. This includes a checkerboard face with two incoming and two outgoing crossings. The endpoint at e_L has one outgoing crossing, that at e_B one incoming crossing, and no other boundary edge changes labels.

There is a directed interface route between these endpoints. Otherwise the set reachable from the first endpoint has no outgoing arc, contradicting its total outdegree-minus-indegree balance. Extract a path and erase loops. Forward dual crossings represent outward primal cut edges and reverse crossings inward edges; only the former contribute to the directed cut. Nonnegativity therefore gives cut cost at least $c_L + c_B + T_N$.

Conversely, a vertex-simple dual corner path, extended through e_L, e_B, is a simple crosscut of the primal square meeting no primal vertex. Planar separation gives a feasible primal cut on the side containing the source boundary arc. Its outward crossings are exactly the forward dual crossings and the two forced edges. Thus its cost is the cost in (1). A minimizing dual walk can be made simple by deleting nonnegative-cost cycles. Apply max-flow/min-cut. For $n = 2$ there is no internal dual step and the same argument gives $V_2 = c_L + c_B$. □

Set $q = 1 - p$ and call a forward dual edge *marked* when its cost is zero. These marks are independent Bernoulli(q). For a simple corner path let k count marked forward steps and b backward steps. Its forward count is $2N + b$, so its cost is $2N + b - k$. Consequently

$$D_N := 2N - T_N = \max_P (k(P) - b(P)), \qquad 0 \le D_N \le 2N. \tag{2}$$

The maximum can be restricted to simple paths. Equivalently every cycle has nonpositive reward, since it has equally many forward and backward steps. Writing $g(q) = 1 - v(1 - q)$, bounded boundary terms in (1) give

$$g(q) = \lim_{N \to \infty} \frac{\mathbb{E} D_N}{2N}. \tag{3}$$

The limit of these expectations exists even without the source's L^1 premise: concatenating optimal paths in consecutive diagonal squares makes $\mathbb{E} D_N$ superadditive, bounded above by $2N$. The elementary superadditive limit lemma applies.

3 The older strict-chain and list estimates

Extend the forward marks independently outside Q_N. Declare a site x open when at least one of its two outgoing forward edges is marked. Different sites use disjoint pairs of edge coordinates. Thus these site indicators are independent with parameter

$$r = 1 - (1-q)^2 = 2q - q^2. \tag{4}$$

Let $A_r(M)$ be the maximum length of a coordinatewise strictly increasing open-site chain in $[0, M)^2 \cap \mathbb{Z}^2$, allowing an open origin. This is BGT's site score $S_r^D((M, M))$ in dimensions $d_1 = 2, d_2 = 0$. Their Lemmas 5, 10 and 13, with the planar Poisson-chain constant identified in their §3.4, give

$$\sigma_r = \lim_{M\to\infty} \frac{\mathbb{E}A_r(M)}{M}, \qquad \lim_{r\downarrow 0} \frac{\sigma_r}{\sqrt{r}} = 2. \tag{5}$$

We state precisely the other consequence of their counting calculation that will be used. This concerns an integer list $w_0 = 0, w_1, \ldots, w_p, w_{p+1} = (M, M)$, with the real intermediate sites distinct, not equal to zero, and strictly below (M, M) in each coordinate. Starting with index zero, repeatedly select the first subsequent site strictly larger in both coordinates. The artificial terminal guarantees termination. Write a for the number of selected gaps and ℓ_j for the number of unselected sites in gap j, $0 \le j < a$. Thus

$$L = \sum_j \ell_j, \qquad p = a - 1 + L.$$

Let $c(j, 0), \ldots, c(j, \ell_j)$ be the selected starting anchor and the intervening sites, and let $c(j + 1, 0)$ denote the next anchor. Set

$$h(j) = c(j+1, 0) - c(j, 0), \qquad u(j, i) = c(j, i) - c(j, i-1).$$

If $\ell_j > 0$, choose a coordinate d_j in which $c(j, \ell_j)_{d_j} \le c(j, 0)_{d_j}$; such a coordinate exists by the first-subsequent selection rule. For an empty gap choose either coordinate. Let d'_j be the other coordinate and define

$$X = \sum_j \sum_{i=1}^{\ell_j} |u(j, i)_{d_j}| + \sum_j \sum_{i=1}^{\ell_j} (u(j, i)_{d'_j})_-. \tag{6}$$

Here $t_- = \max(-t, 0)$. The second term is the *sum of individual negative parts*, as in BGT's (19) and Lemma 19; it is not the negative part of the net displacement.

Lemma 2 (BGT weighted-list consequence)**.** *Fix $\tau > 0$. For all sufficiently small $r > 0$, the sum of the weights r^p over such lists satisfying*

$$c(j, \ell_j)_{d'_j} - c(j+1, 0)_{d'_j} \le 2(\ell_j + 1) \quad (0 \le j < a), \tag{7}$$

$$0 \le -X + 1 + 2L - \tau M\sqrt{r} \tag{8}$$

is at most

$$\frac{C_\tau}{r} \exp(-M\sqrt{r}),$$

for a finite constant C_τ independent of $M \ge 2$.

Identification of the published estimate. Use the $m = 0$ numerical terms of BGT's (33), with $d_1 = 2, d_2 = 0$. Their Items 1 and 3 hold by the telescoping positive anchor increments and the choice of d_j; Item 2 has no coordinates. Their remaining two predicates are (7)–(8). The

weight is exactly $r^{a-1+L} = r^p$. This is a numerical sum over encodings; it does not require a geodesic in the model originally used to derive it.

For clarity, the specialization of their displayed W bound has the following factors. Put $K(t) = (1 + e^{-t})/(1 - e^{-t})$, choose $\beta = 2\gamma$, and set

$$B_0(r) = re^2 e^{2\gamma\sqrt{r}} K(\beta\sqrt{r}) K(\gamma\sqrt{r}),$$
$$B_1(r) = re^{2\gamma\sqrt{r}+2\alpha} e^2 K(\alpha)[K(\gamma\sqrt{r}) + K(\alpha - \gamma\sqrt{r})].$$

For fixed a, (ℓ_j) and (d_j), their coordinate summation is bounded by

$$r^{-1} e^{2+\alpha} e^{(2\beta-\alpha\tau)M\sqrt{r}} B_0(r)^a B_1(r)^L.$$

We have written the gap as $\tau M\sqrt{r}$ in the numerical predicate, thereby fixing its site-parameter normalization. No identification of a bond-FPP event with a site event is used.

Since $K(t) \sim 2/t$ as $t \downarrow 0$, $B_0(r) \to 4e^2/(\beta\gamma)$ and $B_1(r) \to 0$. First choose γ large, then α with $\alpha\tau > 2\beta + 1$, then r small so that $\alpha - \gamma\sqrt{r} > 0$ and $2B_0(r)/(1 - B_1(r)) \le 1/2$. Summing the 2^a coordinate choices and the nonnegative ℓ-vectors gives the geometric series

$$\sum_{a \ge 1} \left(\frac{2B_0(r)}{1 - B_1(r)} \right)^a \le 1.$$

This gives the bound, for example with $C_\tau = e^{2+\alpha}$. These are the already evaluated positive numerical terms of BGT's calculation; enlarging the set of encodings only increases the majorant. □

The meaning of $m = 0$ is important. It prescribes no doubly-open sites in that numerical weight. It does *not* assert that the actual path visits no site with two marked outgoing edges. For any fixed list of distinct sites, requesting that its p sites be open has probability r^p, irrespective of which have two marked edges.

4 The comparison for the asymmetric dual

First, every strict open-site chain counted by $A_r(N)$ can be visited by a monotone path in Q_N that takes a marked outgoing edge at each chain site. Both coordinates increase by at least one between successive sites, so either outgoing edge fits before the next site. Each chosen edge stays inside Q_N, including the final connection. Thus, pathwise,

$$D_N \ge A_r(N). \tag{9}$$

Lemma 3. *For every $\delta > 0$, all sufficiently small fixed $q > 0$ satisfy*

$$\mathbb{P}\{D_N > A_r(N+1) + \delta(N+1)\sqrt{r}\} \le \frac{C_{2\delta}}{r} \exp(-(N+1)\sqrt{r}), \qquad N \ge 1. \tag{10}$$

Proof. Choose a simple minimizing dual path P in Q_N and set $M = N+1$. Every vertex of P is strictly below the auxiliary terminal (M, M). List all open vertices visited by P, excluding its actual terminal (N, N) and treating the origin separately. Append the auxiliary terminal and apply the extraction of Section 3. Let I_0 indicate that the origin is open.

Every marked forward traversal has a distinct visited starting vertex, because P is simple. At most one such traversal leaves each visited vertex, even if both outgoing edges are marked. Therefore

$$k \le p + I_0, \qquad A_r(M) \ge a - 1 + I_0.$$

Together with $D_N = k - b$ and $p = a - 1 + L$, this yields

$$D_N - A_r(M) \le L - b. \tag{11}$$

The origin indicator cancels exactly.

Consider a nonlast gap between two real selected anchors $s \prec t$. Its actual subpath is a minimum-cost path between those anchors: a cheaper replacement would give a cheaper global walk, and nonnegative costs permit subsequent loop removal. A monotone competitor can take a marked first edge if s is open, since both coordinate displacements to t are at least one. Its cost is at most $\|t - s\|_1 - I_s$. The actual segment's cost is $\|t - s\|_1 + b_j - k_j$, with $k_j \le \ell_j + I_s$. Hence

$$b_j \le \ell_j.$$

Any overshoot of the last intervening site beyond t in the other coordinate requires that many backward steps to return. This proves (7), even with ℓ_j in place of $2(\ell_j + 1)$. For the last gap, all real sites are strictly below the auxiliary terminal, so (7) follows directly. No comparison path outside Q_N is needed. Artificially open boundary tails caused by the independent extension only enlarge the upper chain score.

For a gap's distinguished coordinate, the prefix through its last intervening site has nonpositive net displacement. Thus the sum of absolute marked-site increments in this coordinate is at most twice its actual backward variation. In the other coordinate the sum of individual negative increments is at most the corresponding actual backward variation. These prefixes are disjoint in path time across gaps. Consequently the exact quantity (6) satisfies

$$X \le 2b. \tag{12}$$

This factor two is retained, rather than identifying backward costs in two different models.

On the event in (10), (11)–(12) give

$$-X + 2L > 2\delta M\sqrt{r}.$$

In particular (8) holds with $\tau = 2\delta$, including its harmless additional 1. The path therefore supplies a list of distinct open sites meeting all predicates of Lemma 2. For each fixed list its open-site probability is r^p; we may ignore the origin's state. The union bound and that lemma prove (10). There is no conditioning on a chosen geodesic and no independence assumption between different lists. □

To finish Theorem 1, use $0 \le D_N \le 2N$ and $A_r \ge 0$:

$$\mathbb{E}D_N \le \mathbb{E}A_r(N+1) + \delta(N+1)\sqrt{r} + 2N\,\mathbb{P}\{\text{event in (10)}\}.$$

First let $N \to \infty$ at a fixed sufficiently small q. Equations (3), (5) and (9) yield

$$\frac{\sigma_r}{2} \le g(q) \le \frac{\sigma_r}{2} + \frac{\delta\sqrt{r}}{2}.$$

Next let $q \downarrow 0$, using $\sigma_r/\sqrt{r} \to 2$ and $r/(2q) \to 1$. Finally let $\delta \downarrow 0$. This proves the stated asymptotic with the required order of limits.

5 Priority, scope and verification

The entire asymptotic above is a corollary of BGT's older strict-chain and weighted-list estimates after the explicit directed-model event comparison. It is not a new central rare-defect estimate. Our bounded priority checks did not locate an earlier explicit statement of this

particular flow corollary; that fact neither proves first recognition nor establishes that the historical question remained open. This note records the deduction and its attribution without making either assertion.

The ordinary undirected FPP theorem alone is insufficient. In a common box a path's ordinary deficit is $k_f + k_b - 2b$, whereas the present deficit is $k_f - b$. The present deficit minus the ordinary deficit is $b - k_b \geq 0$, unhelpful for the needed upper bound. The larger numerical list estimate, rather than equality of the two models, closes the comparison.

The supporting archive also preserves the original, longer directed-cut counting and BK–Poisson proof and its audits. The Poisson-chain input and upper tail are classical [3]; so is product-space disjoint occurrence [4]. Overlapping-region decomposition by disjoint witnesses and endpoint skeletons also predates this work, including Verges [5, Lemma 2.5 and §5.2]. No originality claim for those general mechanisms is made. The present statement concerns the prescribed edge-capacity model; it makes no assertion for vertex-disjoint paths, alternative boundary conventions or other graph families.

Finite computations test directed duality, cut geometry, sequence constraints and replay of the archived diagnostics. They supplement the analytic proof and the cited estimates, and do not establish the infinite-volume conclusion or novelty. The archive gives exact source fingerprints and bounded-search limitations; third-party publications and raw source corpora are not redistributed.

AI use and review status. AI tools were used extensively in mathematical development, source review, writing, computation and adversarial checking. The recorded adversarial reviews were performed by AI agents. This is an unrefereed preprint; no completed independent human peer review or formal proof-assistant verification is claimed. The author is responsible for the claims.

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