# Multiplicatively Weighted Centers of Median Graphs in Quasilinear Time

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**Abstract**

We record a short reduction from multiplicatively weighted center thresholds to additively weighted eccentricities on a graph with bounded integral distances. Combining it with the algorithm of Bergé, Ducoffe and Habib and a known implicit optimization method gives a deterministic $O(n \log^5(2n))$ algorithm for a center, the optimum radius and all centers of every finite unit-edge median graph. This answers the finite algorithmic form of Open Problem 1 in Ducoffe's Oberwolfach report, without a bound on cube dimension. The contribution is the threshold observation and its consequence; neither the additive algorithm nor the optimization mechanism is new. We specify the exact comparison model and give explicit corrections to two estimates in the public preprint proof of the imported algorithm.

## 1 Problem and prior algorithms

Let $G = (V, E)$ be a finite, nonempty, connected, simple graph with unit edge lengths, $n = |V|$, and shortest-path distance $d$. For a finite nonnegative real profile $w : V \to \mathbb{R}_{\geq 0}$ define

$$r_w(x) = \max_{u \in V} w(u)d(x, u), \qquad \rho = \min_{x \in V} r_w(x), \qquad C_w = \{x : r_w(x) = \rho\}.$$

A median graph is one in which every triple has a unique vertex belonging to the three pairwise shortest-path intervals. Ducoffe [1, pp. 504–505] asks for an almost linear-time weighted-center algorithm on median graphs of bounded cube dimension. The weights in this question multiply distances; additive arrival weights define a different objective.

**Theorem 1.** *Given a graph promised to be median, supplied by adjacency lists, and a finite nonnegative real profile, one can compute $\rho$, a vertex in $C_w$ and the entire set $C_w$ deterministically in $O(n \log^5(2n))$ exact operations. No cube-dimension bound or supplied embedding is required. Operations on profile values only compare bounded integer multiples $kw(u)$, with $0 \leq k \leq n - 1$.*

The principal imported theorem is due to Bergé, Ducoffe and Habib (BDH) [2]: for a nonnegative integer arrival profile $a$, all values

$$E_a(x) = \max_{u \in V} \big(d(x, u) + a(u)\big) \tag{1}$$

on a median graph can be computed in $O(n \log^4(2n))$ time, including preprocessing. The precise interface is on pp. 2–3 of the public preprint [3]; its Algorithm 1 is on p. 22 and the overall

complexity proof is on pp. 23–24. Our calls have $0 \le a(u) \le n-1$. Appendix A supplies consistent parameter and depth estimates for two literal proof errors in that preprint version, preserving the advertised theorem. Classical median structure and its preprocessing algorithms remain credited dependencies [3, 8].

Ducoffe [4, Lemma 2, p. 133:6] already proves that a $T(n,m)$-time weighted-center decision algorithm yields optimization in $O((T(n,m)+n)\log n)$ time. Its weighted-median mechanism is used below in a slightly slower, self-contained comparison-only form. The full preprint statement is Lemma 2.2 on pp. 5–6 [5]. For cube-free median graphs the stronger bound $O(n\log^2 n)$ was already proved by Chalopin et al. [6, Theorem 7.8, p. 28]; our bound does not improve that special case. Tree-product algorithms assume a stronger representation promise [7, 1]. The present note records the short connection between these lines of work. A bounded primary-source audit found no earlier explicit full-scope resolution, but does not certify historical firstness or continued open status in October 2026.

## 2 A threshold identity

Set $D = n-1$. Every distance in a connected unit-edge graph is an integer in $[0, D]$. For $R \ge 0$ and $u \in V$ put

$$b_R(u) = \max\{k \in \{0,\dots,D\} : kw(u) \le R\}, \qquad a_R(u) = D - b_R(u). \tag{2}$$

Zero is an admissible index. If $w(u) = 0$ then $b_R(u) = D$. For positive $w(u)$ this is the capped floor of $R/w(u)$, but binary search on the $D+1$ integer indices computes it using $O(\log(2n))$ comparisons, without a real division or unbounded-floor primitive.

**Lemma 2.** *For every vertex $x$ and $R \ge 0$,*

$$r_w(x) \le R \quad \Longleftrightarrow \quad E_{a_R}(x) \le D.$$

*Proof.* For every $u$, integrality and the distance bound give

$$w(u)d(x,u) \le R \iff d(x,u) \le b_R(u) \iff d(x,u) + a_R(u) \le D.$$

For a zero weight both conditions are vacuous. Take the conjunction over $u$ and use (1). □

Thus one BDH call, followed by a scan, returns the *entire* feasible set $F_R = \{x : r_w(x) \le R\}$ in $O(n\log^4(2n))$ operations. The graph and its zero-weight vertices are retained. No subdivision, Helly property, ball convexity or local-search theorem is required. For $R < 0$ the feasible set is empty. For $n = 1$, or an all-zero profile, the optimum is zero and every vertex is a center.

## 3 Implicit optimization and complete output

For any vertex $x$, its radius is an attained maximum and hence has the form $kw(u)$ with $0 \le k \le D$. Therefore $\rho$ belongs to the multiset of $n$ sorted rows

$$S_u = (0, w(u), 2w(u), \dots, Dw(u)), \qquad u \in V.$$

These $n^2$ entries are conceptual and are never materialized. Here is the promised variant of the prior optimization mechanism [4].

Maintain an inclusive active index interval $[l_u, h_u]$ in each row, initially $[0, D]$, and a feasible incumbent $B = D \max_u w(u)$, witnessed by an arbitrary vertex. Handle $n = 1$ and $B = 0$ directly. Write $L_u = h_u - l_u + 1$ for a nonempty interval and let $M$ be their total length. While $M > 0$, take the lower middle value $q_u = w(u)\lfloor (l_u + h_u)/2 \rfloor$ of every nonempty row, with selection weight $L_u$. Sort these values and choose a weighted median $q$: the total selection weight on either weak side is at least $M/2$. Query $F_q$ using Lemma 2.

If $F_q$ is nonempty, replace $B$ by $\min(B, q)$, saving a corresponding feasible witness when it decreases, and delete all active entries $\geq q$. Otherwise delete all active entries $\leq q$. Binary searches within the row intervals implement either deletion exactly, including ties and zero rows. Repeat until all intervals are empty.

**Lemma 3.** *This process ends after $O(\log(2n))$ queries and returns $B = \rho$ and a center. A final query at $B$ returns exactly $C_w$.*

*Proof.* Maintain that $B$ is feasible and either $B = \rho$ or an active entry equals $\rho$. Initially this holds. A feasible pivot has $\rho \leq q$: if equality holds it is saved; if $\rho < q$, every optimum copy survives. An infeasible pivot has $\rho > q$, so every optimum copy again survives. This proves the invariant, and an empty active set forces $B = \rho$. Its feasible witness is a center, and $F_B = C_w$.

For progress, a middle entry of a row of length $L_u$ has at least $L_u/2$ entries on each weak side. If $q$ is infeasible, rows with $q_u \leq q$ have total length at least $M/2$, and at least half of those entries are deleted. If $q$ is feasible, use rows with $q_u \geq q$. In either case at least $M/4$ entries are deleted, counting multiplicity. This includes singleton rows and equal values. Since $M$ is an integer and starts at $n^2$, only $O(\log(2n))$ iterations occur. □

Sorting and all bounded searches require $O(n \log(2n))$ work per iteration and $O(n)$ row storage beyond the oracle. Thus the total comparison overhead is $O(n \log^2(2n))$. The $O(\log(2n))$ BDH calls and final complete-output call prove Theorem 1. Reading the adjacency lists is absorbed: median graphs have at most $n \log_2 n$ edges [3].

## 4 Model, reproducibility and status

The claim concerns finite vertex centers in a graph promised to be median. It includes zero weights, singleton support, exact equality thresholds and multiple centers. It does not solve recognition, the separate arbitrary-edge-length question, a continuous facility problem, or an unspecified encoding of infinite median graphs. Nor does the global threshold algorithm settle the separate unimodality question in the workshop report [1].

The optimum can be stored as one pair $(k, u)$ representing $kw(u)$. All real comparisons are between such pairs. For rational input $w(u) = p_u/q_u$ with positive denominators and components of at most $B$ bits, a comparison uses integer products $kp_uq_v$ and $jp_vq_u$ of $O(B + \log n)$ bits. There are $O(n \log^2(2n))$ original-weight comparisons; their bit cost and input-reading cost must be paid. The theorem is not an encoding-independent bit bound. The small integer labels in the imported algorithm use $O(\log n)$ bits, including its internal updates, as checked in Appendix A.

The accompanying authored-only verification package includes the frozen original candidate and historical audit, new independent proofs and source comparisons, exact rational controls, the source-proof repair, manuscript source, an offline replay driver and nested hash manifests. Reference all-pairs and brute-structure routines verify finite cases; they are expressly not a fast implementation or an experimental proof of the asymptotic theorem. The universal argument uses the credited classical structure and algorithms. Hashes establish file consistency, not authenticity, priority or formal proof certification.

**AI and review disclosure.** AI tools were used extensively in solving, developing, writing, reproducing and adversarially verifying this work. This is an unrefereed preprint. Internal AI adversarial reviews are not human peer review; the manuscript has not received human peer review or proof-assistant certification. The priority assessment is bounded and the limitations stated above are part of the result's presentation.

# A Explicit estimates for the imported recursion

This appendix concerns the proof of Theorem 1 of *arXiv:2410.10235v1* [3]. It does not assert that the final SODA typesetting contains the same errors; that full chapter was not inspected. BDH's algorithm and structural discoveries remain theirs. The complete step-by-step audit and exact controls are in the supplement.

That preprint declares log = ln on p. 4, proves cube dimension $d \le \log_2 s$, then uses $\ell \le \ln s$ for its wide-ladder size in the duplicated-slice volume estimate on p. 24. Its separate displayed product comparison in Lemma 16 on p. 13 has the wrong direction. Both estimates admit the following consistent replacement.

For every recursive graph of order $s \ge 3$ use

$$L(s) = \lceil \log_2 s \rceil, \qquad f(s) = 2L(s).$$

Compute $L(s)$ as the bit length of $s-1$. A cut is balanced exactly when $f(s)$ times its smaller halfspace size is at least $s$. Keep the original direct base cases $s \le 2$. The structural arguments of BDH's Sections 3–4 apply with this choice: halfspaces and slices are nonempty gated median graphs; the unique-majority median in an unbalanced instance needs only $f(s) \ge 2$. The wide ladder consists of pairwise orthogonal incident classes, so $\ell \le \log_2 s \le L(s)$. Each slice $S_i$ lies in a minority halfspace of size less than $s/f(s)$. The two recursive calls on each slice consequently have total volume

$$2\sum_{i=0}^{\ell-1} |S_i| \le \frac{2\ell s}{f(s)} \le s.$$

Balanced children partition $s$. Each child in either case has order at most $s(1-1/f(s))$; in the unbalanced case its order is less than $s/f(s)$. The gate, fiber-reweighting and merging identities depend on gatedness, not this balance constant. Both original-weight and fiber-weight slice calls are retained: a fiber weight need not dominate the original value. There is no change of objective or input graph.

For completeness, put $\lambda = 4/\ln 2$. Since $f(s) \le 4\log_2 s = \lambda \ln s$ for $s \ge 3$, every nonterminal path from a root of order $N$ satisfies

$$s_{i+1} \le s_i \left(1 - \frac{1}{\lambda \ln s_i}\right) \le s_i \left(1 - \frac{1}{\lambda \ln N}\right).$$

Thus $s_h \le N \exp(-h/(\lambda \ln N))$. At $h = \lceil \lambda (\ln N)^2 \rceil$ this is at most one, so the path has already reached the $s \le 2$ base case. This proves depth $O(\log^2 N)$ directly, without the erroneous product comparison. The source's nonrecursive work is $O(s \log^2(2s))$; per-level volume is at most $N$. The total remains $O(N \log^4(2N))$.

For our initial offsets at most $N-1$, each fiber update adds at most $N-1$ to a previous maximum. Along $O(\log^2 N)$ levels all internal arrival labels are $O(N \log^2(2N))$, hence have $O(\log(2N))$ bits. The cited linear-time median/Theta/halfspace preprocessing [8] is unchanged. It is not reconstructed as a fast implementation here.

**Why the literal volume estimate needs correction.** Take $Q_{30}$, attach 17,000,000,000 leaves at each of its 30 unit-vector vertices and 488,926,258,176 leaves at zero. This is a median graph of order $N = 10^{12}$: attaching pendant trees at a vertex preserves medianity. Each coordinate minority has 17,536,870,912 vertices. A finite positive Taylor sum gives $e^7 > 1000$, hence $\ln N < 28$; multiplying each such minority by 56 gives $982{,}064{,}771{,}072 < N$. All cuts are unbalanced for the written natural-log threshold $2 \ln N$. Give the cube antipode arrival weight $N - 1$ and every other vertex zero. It is uniquely farthest from the majority median zero, with all 30 classes in its wide ladder. Their minority union has 511,073,741,823 vertices; its two slice copies therefore total $1{,}022{,}147{,}483{,}646 > N$. This succinct exact example refutes the intermediate volume estimate, not correctness or the final time bound after the stated repair.

## References


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