

A Minimal-Surface Counterexample to a Seshadri Lower Bound

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Abstract

We give an explicit family of counterexamples to the proposed lower bound $\varepsilon(S, L; x) \geq (2 + |K_S^2|^{1/4})^{-1}$ for minimal complex projective surfaces. For every integer $d \geq 7$, a classical plane-pencil construction followed by a double base change gives a smooth surface with ample canonical bundle, $K_S^2 = 6(d-1)(d-3)$, and an ample integral line bundle L with $L^2 = 6$ and $\varepsilon(S, L; x) \leq 1/(d-1)$. Already $d = 7$ contradicts the bound. The pencil and covering ingredients are classical; historical priority has not been fully cleared. This unrefereed note used AI tools extensively and has not received human peer review.

1 The formulation and the result

For a smooth complex projective surface S , an ample integral line bundle L , and a closed point x , put

$$\varepsilon(S, L; x) = \inf_{\substack{C \subset S \\ \text{integral curve} \\ x \in C}} \frac{L \cdot C}{\text{mult}_x C}.$$

Szemberg's final question in the 2007 preprint of [1] asks for a minimal polarized surface violating

$$\varepsilon(S, L; x) \geq \frac{1}{2 + |K_S^2|^{1/4}}. \quad (1)$$

It is restated as Conjecture 2 in his Oberwolfach contribution [2, p. 1124] and as Question 6.1.6 in [3, p. 17], using the cited preprint version for that locator. The point is arbitrary, and there is no Picard-rank restriction. The additive 2 lies outside the fourth root; the problem catalogue's different transcription is not the primary formula.

Theorem 1. *For every integer $d \geq 7$ there are a smooth integral complex projective surface S , an ample integral line bundle L on S , and a point $x \in S$ such that K_S is ample and*

$$K_S^2 = 6(d-1)(d-3), \quad L^2 = 6, \quad \varepsilon(S, L; x) \leq \frac{1}{d-1} < \frac{1}{2 + (K_S^2)^{1/4}}.$$

In particular S is minimal of general type and (1) is false.

The starting pencil and its polarization are Miranda's construction, recorded by Ein–Lazarsfeld [4, Example 3.1, p. 182], Bauer [5, Proposition 3.3 and its proof], and Bauer–Szemberg [6, Proposition 2.1, p. 2]. We use a double base change branched at four smooth fibres. We claim neither newness of these operations nor historical firstness of this counterexample. Section 5 gives the priority and verification limits.

2 An integral pencil and an ample line bundle

We supply the existence argument at finite degree, rather than relying on the asymptotic $d = m + 1$ assertion in [5, Proposition 3.3(b)]. In $\mathbb{P}^2$ use coordinates $[X : Y : Z]$ and set

$$f_d = Z(X^{d-1} - Y^{d-1}) + X^d, \quad C_0 = V(f_d), \quad p = [0 : 0 : 1].$$

The coefficients of this linear polynomial in Z are coprime in $\mathbb{C}[X, Y]$. Gauss's lemma, applied over $\mathbb{C}(X, Y)$, proves irreducibility. In the chart $Z = 1$, the tangent cone at p is $x^{d-1} - y^{d-1}$, a product of distinct linear factors. Thus p is an ordinary $(d - 1)$ -fold point. It is the only singular point: the equations $f_Z = f_Y = 0$ force either $X = Y = 0$, or $X, Y \neq 0$ and $Z = 0$; in the latter case $f_X = dX^{d-1} \neq 0$.

Write $N_e = e(e + 3)/2$. The reducible or nonreduced degree- d locus R in $\mathbb{P}^{N_d}$ is the finite union of the closed images of multiplication maps

$$\mathbb{P}^{N_a} \times \mathbb{P}^{N_{d-a}} \longrightarrow \mathbb{P}^{N_d}, \quad 1 \leq a \leq d - 1.$$

Since $N_d - N_a - N_{d-a} = a(d - a) \geq d - 1$, its dimension is at most $N_d - d + 1$. The closure of the union of lines joining $[f_d]$ to R has dimension at most $N_d - d + 2 < N_d$ for $d \geq 3$. Choose a form g outside that closure. Then the entire pencil $\langle f_d, g \rangle$ avoids R . We can simultaneously choose g smooth, nonzero at p , and transverse to the smooth locus of C_0 : these are nonempty open conditions in the same irreducible parameter space, by Bertini and transversality for the complete degree- d system. Bézout gives exactly d^2 distinct simple base points p_i , all away from p . Thus every pencil member is integral and reduced.

Blow up these base points once each, obtaining $b : Y \rightarrow \mathbb{P}^2$. Let $H = b^*\mathcal{O}_{\mathbb{P}^2}(1)$, and let E_i be the exceptional curves. Transversality resolves the pencil and gives $f : Y \rightarrow \mathbb{P}^1$, with fibre class

$$F = dH - \sum_{i=1}^{d^2} E_i.$$

All fibres are strict transforms of integral reduced pencil members; the E_i are sections, not vertical components. General fibres are smooth. The special fibre C , the strict transform of C_0 , still has multiplicity $d - 1$ at the point y above p .

Put $A = 2F + E_1$. The blowup intersection rules give

$$F^2 = 0, \quad F \cdot E_i = 1, \quad E_i^2 = -1, \quad A^2 = 3, \quad A \cdot E_1 = 1, \quad A \cdot F = 1.$$

Here $F = f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is nef. For every integral curve $D \neq E_1$, $E_1 \cdot D \geq 0$. If $F \cdot D > 0$, then $A \cdot D > 0$. If $F \cdot D = 0$, D is vertical, hence is an entire fibre because each fibre is integral and reduced; in this case $A \cdot D = 1$. Together with $A \cdot E_1 = 1$ and $A^2 > 0$, the Nakai–Moishezon criterion proves that A is ample. The assertion about every fibre is essential here.

3 The cover and minimality

Choose four distinct smooth fibre values of f , avoiding the value of C . Let $h : B \rightarrow \mathbb{P}^1$ be the connected double cover simply branched at these values. For example its equation in a coordinate avoiding the four values at infinity is $v^2 = \prod_{j=1}^4 (t - t_j)$. Its smooth projective completion is connected of genus one by Hurwitz. Form the fibre product

$$S = Y \times_{\mathbb{P}^1} B, \quad \pi : S \rightarrow Y,$$

so π is finite flat of degree two. It is étale off the four branch fibres, including above the singularities of all other fibres. Along a branch fibre, f is smooth and the local cover has equation $t = v^2$ in smooth coordinates (u, t) on Y , so S is smooth there as well. The map $S \rightarrow B$ is proper with connected fibres, because they are the fibres of f . Since B is connected, S is connected: a decomposition into two open and closed subsets would give a decomposition of B , using properness and connected fibres. A connected smooth scheme has a single irreducible component here, since distinct components of a smooth scheme cannot meet. Hence S is integral. It is projective as a fibre product of projective varieties.

The branch divisor on Y is linearly equivalent to $4F$. The double cover is associated with $\mathcal{O}_Y(2F)$. Hurwitz therefore gives the equality of line bundles

$$K_S = \pi^*(K_Y + 2F). \quad (2)$$

This is the classical cyclic-cover formula; compare [7, Section 2, p. 3]. In the present local model the ramification divisor has line bundle $\pi^*\mathcal{O}_Y(2F)$ and is reduced, also giving (2) directly. Since $K_Y = -3H + \sum E_i$, put

$$N = K_Y + 2F = (d-3)H + F.$$

We have $N \cdot E_i = 1$, and for every other integral curve D , its image in $\mathbb{P}^2$ is a curve of degree $e > 0$, so

$$N \cdot D = (d-3)e + F \cdot D > 0 \quad (d \geq 4).$$

Moreover $H^2 = 1$ and $H \cdot F = d$, whence

$$N^2 = (d-3)^2 + 2d(d-3) = 3(d-1)(d-3) > 0.$$

Nakai–Moishezon proves N ample. A finite surjective pullback of an ample line bundle is ample (alternatively apply the projection formula and Nakai on S). Thus (2) proves K_S ample. In particular S is of general type and has no rational (-1) -curve, since adjunction would give canonical degree -1 on such a curve. Consequently S is minimal.

4 The pointed curve and the comparison

Set $L = \pi^*A$, an ample integral line bundle. The degree-two intersection formula gives

$$L^2 = 2A^2 = 6, \quad K_S^2 = 2N^2 = 6(d-1)(d-3).$$

The fibre value of C is unbranched. Its two inverse images under h are distinct complex points, so

$$\pi^{-1}(C) = C' \sqcup C'', \quad C' \simeq C \simeq C''.$$

In particular this is not a connected degree-two curve over C . Choose $x \in C'$ mapping to y . The local isomorphism, or étaleness and the split fibre identification, gives $\text{mult}_x C' = d-1$. Projection gives $L \cdot C' = A \cdot C = 1$. Therefore

$$\varepsilon(S, L; x) \leq \frac{L \cdot C'}{\text{mult}_x C'} = \frac{1}{d-1}.$$

For $d \geq 7$ the desired strict inequality is equivalent, after subtracting 2 and taking fourth powers of positive numbers, to

$$(d-3)^3 > 6(d-1).$$

At $d = 7$ this is $64 > 36$. The forward difference of the left side minus the right side is $3(d - 3)^2 + 3(d - 3) - 5 > 0$ for $d \geq 7$, proving the claim for every such integer. In the smallest example $K_S^2 = 144$, $L^2 = 6$, and $144 < 4^4$ yields $1/6 < (2 + 144^{1/4})^{-1}$. This proves Theorem 1.

Only the displayed upper bound is asserted; an exact Seshadri value is unnecessary. The surfaces vary with d , and x is a special point. The construction does not resolve positivity of the infimum over all ample integral line bundles and points on one fixed complex surface. It does not contradict very-general-point or Picard-rank-one lower bounds, or canonical and adjoint polarization bounds. For $d = 4, 5, 6$ this particular curve does not give the strict comparison; this says nothing about other curves or lower bounds on those surfaces.

5 Attribution, priority and verification limits

Miranda’s pencil polarization is classical, with the direct record in [4], its extension in [5], and the close arbitrary-rational construction in [6]. Those examples use nonminimal blowups. Earlier double-cover examples in [8, Example 1.3] concern the canonical polarization and produce a quotient $1/2$; that quotient cannot violate (1). Kollár’s example in [4, Example 3.7] concerns unbounded indices of very ampleness, which is a different assertion. Fuentes García [7] treats cyclic-cover constants in other settings, including very general points over Picard-rank-one bases and points on ramification divisors. The present curve is over an unbranched singular fibre of a plane blowup.

Two independent internal primary-source priority audits, supplemented by the author’s source comparisons, found no explicit earlier resolution of this precise bound in the text inspected. Historical priority nevertheless cannot be fully cleared. In particular the full Bansal–Majumder 2025 survey [9], the full Roy 2019 thesis, and some other theses were not inspected; longer sources and later citation bodies were partly or selectively read, and final journal versions were not globally compared. The verification package records exact reading scopes, access limits, fingerprints and search ledgers. We make no claim of first discovery, novelty of the covering method, exhaustive priority clearance, or independently verified openness immediately before this note.

The accompanying authored-only verification package preserves the original 18 inputs from PR 585, new independent derivations and adversarial reviews, source receipts, exact symbolic controls, and pinned nested inventories. Its replay requires preinstalled SymPy 1.14.0 and mpmath 1.3.0; raw third-party documents and account material are excluded. The finite controls supplement the geometric proof, including its Bertini and Nakai–Moishezon inputs; they do not replace existence arguments or establish priority. Integrity checks do not certify authenticity or semantic authorship.

AI and review disclosure. AI tools were used extensively in solving, research, writing, source comparison, reproduction and adversarial verification. Reviews described here are internal AI reviews. This is an unrefereed preprint: it has not received human peer review and is not certified by a proof assistant.

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