An elementary counterexample to Long's clique equality

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17 October 2026

Abstract

For the full ring of integers of $\mathbb{Q}(\sqrt{10})$, both the unit-difference graph and the graph allowing unit or prime-element differences have clique number 2, while the least norm of a nonzero prime ideal is 2. Thus the two sides of the equality asked about in Long's Question 7.2 in arXiv:1512.04620v1 are 2 and 4. The proof uses the classical nonprincipal ideal $(2, \sqrt{10})$ and its two-element residue field. We include the elementary arithmetic and specify the prime-element convention, which is essential to the conclusion. No novelty or historical priority claim is made.

1 The precise question

Question 7.1 of [1, p. 5] starts with an irreducible $f \in \mathbb{Z}[X]$ all of whose roots are real, puts $k = \mathbb{Q}(\alpha)$ for a root α, and takes $\mathcal{O}_k$ to be the full ring of integers. It asks how often $\mathcal{O}_k$ is a principal ideal domain; it does not impose that property as a hypothesis. Question 7.2 [1, p. 6] defines Γ_{unit} on $\mathcal{O}_k$ by unit differences and enlarges it to Γ_{prime} by allowing prime differences. Its displayed formula is

$$|\operatorname{clique}(\Gamma_{\text{prime}})| \le |\operatorname{clique}(\Gamma_{\text{unit}})| \min\{\, |\mathcal{O}_k/I| \mid I \text{ prime} \,\}, \tag{1}$$

followed by the question "Is this an equality?" We write $\omega(G)$ for the clique number denoted by $|\operatorname{clique}(G)|$ there.

Here a *prime element* is a nonzero nonunit p such that (p) is a prime ideal. This convention is fixed by the manuscript cited as reference [19] in the problem list: condition (i) of its Definition 1 requires a difference to be a unit or to generate a prime ideal, and Section 3 defines prime cliques using condition (i) alone [2, pp. 1, 9]. The published Long–Thistlethwaite framework retains these definitions [3, Definition 1, p. 565; Section 3, p. 572]. The additional condition about surjectivity from the unit group belongs to Lenstra–Hurwitz cliques and is not imposed on Γ_{prime}.

The minimum in (1) is over prime ideals. In contrast, Theorem 3.1 of [3, p. 573] is printed with a minimum *principal*-ideal norm. We do not identify these minima or import that theorem: the proof below evaluates each quantity in the question's display directly.

Theorem 1. *Let $k = \mathbb{Q}(\sqrt{10})$ and let $O = \mathcal{O}_k$. With the definitions above,*

$$\omega(\Gamma_{\text{unit}}) = \omega(\Gamma_{\text{prime}}) = 2, \qquad \min_{\substack{\mathfrak{q} \text{ nonzero} \\ \text{prime ideal of } O}} |O/\mathfrak{q}| = 2.$$

Consequently the proposed equality in Long's Question 7.2 is false:

$$\omega(\Gamma_{\text{prime}}) = 2 < 4 = \omega(\Gamma_{\text{unit}}) \min_{\mathfrak{q} \text{ nonzero prime}} |O/\mathfrak{q}|.$$

The ingredients are classical. Lenstra's residue argument already bounds a unit clique by the cardinality of any proper ideal quotient [4, Proposition 2.2, p. 242]. Dummit's 2021 notes explicitly give the ideal $(2, \sqrt{10})$, its nonprincipality, and the obstruction to norm ± 2 modulo 5 [5, p. 13]. These are credited ingredients of the following self-contained calculation, rather than claims of a new unit bound or new ideal arithmetic.

2 The arithmetic and the two colors

The polynomial $X^2 - 10$ is irreducible over $\mathbb{Q}$: a rational root of this monic polynomial would be an integer with square 10. Its two roots are real, so this field satisfies the setup of Question 7.1.

Lemma 1. *The full ring of integers is $O = \mathbb{Z}[\sqrt{10}]$. For every nonzero $\delta = a + b\sqrt{10} \in O$,*

$$[O : \delta O] = |a^2 - 10b^2|. \tag{2}$$

Proof. Every $a + b\sqrt{10}$ with $a, b \in \mathbb{Z}$ is a root of the monic polynomial $X^2 - 2aX + (a^2 - 10b^2)$ and is integral. Conversely, suppose $r + s\sqrt{10}$ is integral, with $r, s \in \mathbb{Q}$. Its trace $2r$ and norm $r^2 - 10s^2$ are integers. For $s \neq 0$ this follows from the coefficients of its monic minimal polynomial (by Gauss's lemma); for $s = 0$ a rational algebraic integer is an integer.

Put $m = 2r \in \mathbb{Z}$. Since $40s^2 = m^2 - 4(r^2 - 10s^2) \in \mathbb{Z}$, the reduced denominator v of s satisfies $v^2 \mid 40$. Thus $v = 1$ or 2, and $n = 2s \in \mathbb{Z}$. Integrality of the norm now gives

$$m^2 - 10n^2 \equiv 0 \pmod 4.$$

An odd n would require a square to be 2 modulo 4. Hence n is even, and then m is even too. Therefore $r, s \in \mathbb{Z}$.

In the integral basis $(1, \sqrt{10})$, multiplication by δ has matrix

$$M_\delta = \begin{pmatrix} a & 10b \\ b & a \end{pmatrix}.$$

Its determinant is $a^2 - 10b^2 \neq 0$, since $\delta \neq 0$ and $\sqrt{10} \notin \mathbb{Q}$. Its image is δO. The index of the image of a nonsingular integer matrix is the absolute value of its determinant: unimodular row and column operations reduce to Smith normal form, where this assertion is immediate. This proves (2). □

Define the surjective ring homomorphism

$$\varphi : O \longrightarrow \mathbb{F}_2, \qquad a + b\sqrt{10} \longmapsto a \pmod 2.$$

Multiplicativity follows because the constant coefficient of a product is $ac + 10bd$. The kernel is

$$P = (2, \sqrt{10}), \qquad O/P \cong \mathbb{F}_2.$$

Indeed, a zero residue means that the constant coefficient is even. Thus P is a nonzero proper maximal prime ideal of norm 2.

This ideal is nonprincipal. If $P = (a + b\sqrt{10})$, its index and (2) would give $a^2 - 10b^2 = \pm 2$. Modulo 5 this requires $a^2 \equiv 2$ or 3, whereas the squares are $0, 1, 4$. This is impossible.

We next verify the maximality fact needed to exclude prime elements from P. Let Q be any nonzero prime ideal of O, and choose $0 \neq \delta \in Q$. By (2), $O/(\delta)$ is finite, so its quotient O/Q is a

finite integral domain. Such a domain is a field: multiplication by a nonzero element is injective, hence surjective, and therefore has 1 in its image. It follows that every nonzero prime ideal of O is maximal.

No unit lies in the proper ideal P, since otherwise $1 \in P$. If a prime element p lay in P, the nonzero prime ideal (p) would be maximal and satisfy $(p) \subseteq P \subsetneq O$. Hence $(p) = P$, contrary to nonprincipality. Thus P contains neither units nor prime elements.

Proof of Theorem 1. If two vertices have the same image under φ, their difference lies in P and cannot be an allowed edge difference in either graph. The two residue classes therefore give a proper coloring of both graphs with two colors. A clique has at most one vertex of each color, so both clique numbers are at most 2. The vertices 0 and 1 are adjacent in both graphs, giving equality.

Every proper ideal quotient has at least two elements, since $0 \neq 1$ in that quotient. The prime ideal P attains this lower bound. Therefore the minimum nonzero prime-ideal norm is 2, and the two sides of the proposed equality are 2 and $2 \cdot 2 = 4$. If the unqualified word "prime" in (1) includes the zero ideal, its quotient is the infinite ring O and cannot lower this minimum. □

3 Conventions, attribution, and verification

The prime-element convention matters. The element 2 is not prime: $\sqrt{10} \notin (2)$ but $(\sqrt{10})^2 = 10 \in (2)$. It is nevertheless irreducible. Indeed, its absolute norm is 4, absolute norm 1 is equivalent to being a unit (use conjugation for the inverse), and a factorization into nonunits would require factors of absolute norm 2, which the congruence above excludes. Consequently the graph that allows irreducible differences contains the triangle $\{0, 1, 2\}$ and is a different problem.

Lopez's first prime-power graph $\mathcal{P}_k^{(1)}$ is exactly the prime-element graph used here [6, Definition 52, p. 26]. For $n \geq 2$, the graph $\mathcal{P}_k^{(n)}$ also permits differences whose principal ideals are higher powers of prime ideals. In this example

$$P^2 = (4, 2\sqrt{10}, 10) = (2) :$$

all displayed generators are multiples of 2, and $2 = 10 - 2 \cdot 4$. Thus $\{0, 1, 2\}$ is a triangle in every $\mathcal{P}_k^{(n)}$ with $n \geq 2$, although 2 is not a prime element. Lopez's bound 4 in the ramified case [6, Theorem 55, p. 27] concerns $\mathcal{P}_k^{(1)}$ and is consistent with its exact value 2 here. The ramification clause just cited is the thesis's wording; the journal Corollary 3.1.1 [3, p. 573] literally states its 4-element bound for the case where 2 splits.

The conclusion concerns the equality exactly printed in [1, Question 7.2], with the full ring of integers and no principal-ideal-domain premise. It makes no assertion about a separately restricted PID version. The preceding citations record the definitions and classical ingredients used here. We make no claim of novelty, firstness, historical priority, or current worldwide openness of the printed question.

An accompanying verification package records source receipts and exact finite diagnostic checks of the elementary arithmetic and boundary conventions. Those computations are supplementary; the proof above is complete and does not infer universal claims from a finite search. Third-party primary-source PDFs are cited rather than redistributed.

Disclosure. AI tools were used extensively in solving, verifying, and writing this note. The note is unrefereed, has received no external human peer review, and has no formal proof-assistant certification.

References

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[5] E. Dummit, *Number Theory (part 9): The Geometry of Numbers*, 2021, version 1.90, p. 13 (the $\mathbb{Z}[\sqrt{10}]$ example). `https://dummit.cos.northeastern.edu/teaching_sp21_4527/numthy_9_geometry_of_numbers_v1.90.pdf`.

[6] D. S. Lopez, *Lenstra-Hurwitz Cliques In Real Quadratic Fields*, M.S. thesis, San José State University, 2018, Definition 52, p. 26; Theorem 55, p. 27. doi:10.31979/etd.3bc5-77v2.