Local Ricci-pinching maxima on an almost-abelian group

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Abstract

Lauret's July 2018 talk announced nonglobal local maxima of the Ricci pinching functional that are not solvsolitons of the specified transitive group. This note gives a self-contained small-parameter proof for the known Lauret–Will family $A_u = J \oplus uE_{12}$. For $0 < u < 1/2$, its standard metric is a non-strict local maximum of $F = \mathrm{Scal}^2 / \|\mathrm{Ric}\|^2$, with value $1/3$, among all nearby left-invariant metrics on its fixed five-dimensional simply connected group. The metric fails that group's equation $\mathrm{Ric} = cI + D$, $D \in \mathrm{Der}(\mathfrak{g})$, and a same-group metric has value $9/17$. The proof uses a full orthogonal slice and a uniform remainder estimate along an exact constant-value family. The metric is nevertheless a Ricci soliton, isometric to $\mathbb{R}^2 \times H_3$. No novelty, first assertion, or first proof is claimed. This AI-assisted research note is unrefereed.

1 Statement and source context

The existence statement addressed here was already announced by Jorge Lauret on 23 July 2018, in a displayed theorem attributed to Lauret–Will [2, slide 13/17, overlay PDF pp. 71–73]. It asserts the existence of Ricci-soliton local maxima that are nonglobal and are not solvsolitons. The checked slides do not supply a construction or proof of that clause. The exact family used below, denoted D_t, and its value $1/3$ are in Lauret–Will [3, §6.4, p. 19] and [4, §6.4, p. 20]. Their critical-point classification and nonglobality results are also prior work [4, Theorem 1.5, Theorem 6.5(iii), Proposition 6.11].

Both checked arXiv versions explicitly leave small nilpotent coefficients unresolved immediately after Lemma 6.12 [3, pp. 18–19][4, p. 19]. These statements postdate the talk announcement. This note does not infer the reason for their difference. The full journal text was not retrieved, so the statements about passages and page numbers concern the specified arXiv versions. The proof below stands on its own; neither the source chronology nor a claim of present community-open status is a premise.

Question 14 in §11 of the Morgan–Pansu collection [1, p. 12], proposed by Lauret, asks whether local maxima of F among left-invariant metrics on a fixed solvable group are solvsolitons. In that question a solvsoliton means

$$\mathrm{Ric}_g = cI + D, \qquad c \in \mathbb{R}, \quad D \in \mathrm{Der}(\mathfrak{g}), \tag{1}$$

where $\mathfrak{g}$ is the Lie algebra of the specified group. Thus the group presentation is part of the assertion. Here and throughout, Ric_g denotes the Ricci endomorphism and its norm is the Hilbert–Schmidt norm with respect to g. Local maximality uses the usual topology on the cone of positive definite inner products on $\mathfrak{g}$.

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and let E_{12} be the 2 by 2 matrix with its only nonzero entry in position $(1,2)$. For $u > 0$, define

$$A_u = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{pmatrix} = J \oplus uE_{12}.$$

Let $\mathfrak{g}_u = \mathbb{R}e_0 \oplus V$, with $V = \operatorname{span}(e_1, e_2, e_3, e_4)$, have bracket $[e_0, X] = A_u X$ and $[V, V] = 0$. This is a solvable Lie algebra: its derived algebra is contained in the abelian ideal V. Write G_u for its simply connected group and g_u for the metric making the displayed basis orthonormal.

Theorem 1. *For each fixed $0 < u < 1/2$, the metric g_u is a non-strict local maximum of F among all left-invariant metrics on G_u, and $F(g_u) = 1/3$. It does not satisfy (1) on $\mathfrak{g}_u$. It is not a global maximum: a left-invariant metric on the same group has $F = 9/17$.*

This supplies a negative answer to the literal fixed-group assertion in Question 14, already announced in the talk cited above. It is compatible with the Ricci-soliton local-maximum conclusion of Lauret–Will [4, Theorem 1.5]. Section 6 explains the metric's Ricci-soliton presentation explicitly. No conclusion for completely solvable groups or groups of real type follows from this example.

2 Curvature and the space of metrics

We first make the curvature computation and the all-metrics reduction explicit. For an arbitrary real matrix $A : V \to V$, consider the orthonormal almost-abelian presentation $[e_0, X] = AX$. Put

$$S = \tfrac{1}{2}(A + A^T), \qquad K = \tfrac{1}{2}(A - A^T).$$

For left-invariant $X, Y \in V$, the Koszul formula gives

$$\begin{aligned} \nabla_{e_0} e_0 &= 0, & \nabla_{e_0} X &= KX, \\ \nabla_X e_0 &= -SX, & \nabla_X Y &= \langle SX, Y \rangle e_0. \end{aligned} \tag{2}$$

Use the convention $R(U,W) = \nabla_U \nabla_W - \nabla_W \nabla_U - \nabla_{[U,W]}$. Substituting (2) yields

$$R(e_0, X)e_0 = (S^2 + [S, K])X, \qquad R(X, Y)Z = -\langle SY, Z \rangle SX + \langle SX, Z \rangle SY.$$

Taking the Ricci trace gives

$$\operatorname{Ric}_{00} = -\operatorname{tr} S^2, \qquad \operatorname{Ric}_{0V} = 0, \qquad \operatorname{Ric}|_V = [K, S] - (\operatorname{tr} A)S = \tfrac{1}{2}[A, A^T] - (\operatorname{tr} A)S. \tag{3}$$

For example, the e_0 contribution to $\operatorname{Ric}|_V$ is $-S^2 - [S, K]$, and the trace over V is $S^2 - (\operatorname{tr} S)S$, giving the stated expression. This also recovers the almost-abelian formula in [4, equation (3)].

Every matrix used below has trace zero. With the Frobenius norm, put

$$s(A) = \|S\|^2, \qquad q(A) = \|[A, A^T]\|^2.$$

Equation (3) implies

$$\operatorname{Scal} = -s, \qquad \|\operatorname{Ric}\|^2 = s^2 + \tfrac{1}{4}q, \qquad F(A) = \frac{s^2}{s^2 + q/4}. \tag{4}$$

When $s > 0$, this reduces the desired inequality to

$$F(A) \le \tfrac{1}{3} \quad \Longleftrightarrow \quad P(A) := q(A) - 8s(A)^2 \ge 0. \tag{5}$$

For A_u, direct multiplication gives $s(A_u) = u^2/2$, $q(A_u) = 2u^4$, and hence $F(A_u) = 1/3$. The function F is invariant under nonzero scalar multiplication of A and under orthogonal conjugation.

Fix $u = u_0$ in the theorem's range. A nearby inner product on the fixed vector space $\mathbb{R}e_0 \oplus V$ has positive definite Gram matrix

$$\Gamma = \begin{pmatrix} \alpha & \beta^T \\ \beta & \Gamma_V \end{pmatrix}.$$

Set

$$T = \Gamma_V^{-1/2}, \qquad c = (\alpha - \beta^T \Gamma_V^{-1} \beta)^{-1/2}.$$

The columns of T give an orthonormal basis of V, and $f_0 = c(e_0 - \Gamma_V^{-1}\beta)$ is its unit orthogonal complement. The component of f_0 in V has zero adjoint action on V. Thus the bracket matrix in this orthonormal basis is exactly

$$cT^{-1} A_{u_0} T. \tag{6}$$

As $\Gamma \to I$, both $c \to 1$ and $T \to I$ smoothly. Consequently it suffices to prove (5) for every conjugate of A_{u_0} sufficiently close to A_{u_0}. Conversely, any conjugate is realized by a metric on the fixed group: an arbitrary invertible change of basis in V, declared orthonormal with e_0, gives such a matrix. Cross terms in Γ have therefore been fully included in the local reduction.

3 A full orthogonal slice

Let A be a nearby conjugate of A_{u_0}. Its generalized zero eigenspace

$$W = \ker A^2$$

is two-dimensional and invariant. The relation $A^4 + A^2 = 0$ implies that $E = I + A^2$ is an idempotent with image W. This is a smooth projection, generally not an orthogonal one. We can construct a smooth local orthonormal frame directly: put $p = Ee_4$ and

$$f_3 = \frac{Ap}{\|Ap\|}, \qquad f_4 = \frac{p - \langle p, f_3\rangle f_3}{\|p - \langle p, f_3\rangle f_3\|}.$$

At A_{u_0} the denominators are u_0 and 1, respectively, so neither vanishes nearby. Since $A^2 p = 0$, these choices give

$$Af_3 = 0, \qquad Af_4 = v f_3, \qquad v = \frac{\|Ap\|}{\|p - \langle p, f_3\rangle f_3\|} > 0, \quad v \longrightarrow u_0.$$

Project e_1, e_2 orthogonally onto $W^\perp$ and apply Gram–Schmidt to obtain a smooth frame f_1, f_2 close to e_1, e_2.

In the frame f_1, f_2, f_3, f_4, invariance of W gives

$$M = \begin{pmatrix} C & 0 \\ L & N \end{pmatrix} = \begin{pmatrix} a & b-k & 0 & 0 \\ b+k & -a & 0 & 0 \\ x & y & 0 & v \\ z & w & 0 & 0 \end{pmatrix}, \qquad k = \sqrt{1 + a^2 + b^2}. \tag{7}$$

Indeed, C represents the quotient action on V/W, which has eigenvalues $i, -i$. It has trace zero and determinant one. Writing its symmetric part as $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and skew part as kJ gives $k^2 - a^2 - b^2 = 1$; proximity to J chooses $k > 0$. All six transverse coordinates $\xi = (a, b, x, y, z, w)$ tend to zero as $A \to A_{u_0}$. The lower-left block L is unrestricted in the estimates that follow. This point is essential: $W^{\perp}$ need not be invariant.

For a checkable expansion, block multiplication gives

$$[M, M^T] = \begin{pmatrix} [C, C^T] - L^T L & CL^T - L^T N \\ LC^T - N^T L & LL^T + [N, N^T] \end{pmatrix}.$$

It follows that

$$q(M) = \|[C, C^T] - L^T L\|^2 + 2\|LC^T - N^T L\|^2 + \|LL^T + [N, N^T]\|^2, \tag{8}$$

$$s(M) = 2(a^2 + b^2) + \tfrac{1}{2}(x^2 + y^2 + z^2 + w^2 + v^2). \tag{9}$$

In particular, $s(M) \geq v^2/2 > 0$. Put $h^2 = a^2 + b^2$, $r^2 = x^2 + y^2 + z^2 + w^2$, and $d = wx - yz$. Expanding (8)–(9), using $k^2 = 1 + h^2$, gives the exact identity

$$P(M) = Q_v(\xi) + R_v(\xi), \tag{10}$$

where

$$\begin{aligned} Q_v(\xi) = {} & 16(2 - v^2)(a^2 + b^2) + 2(x + vw)^2 + 2(y - vz)^2 \\ & + 2(1 - 4v^2)(z^2 + w^2) \end{aligned} \tag{11}$$

and

$$\begin{aligned} R_v(\xi) = {} & -12h^2 r^2 - 24ak(wz + xy) + 12bk(x^2 + z^2 - y^2 - w^2) \\ & + 4v\{a(wy - xz) - b(wx + yz)\} + 4v(k - 1)d - 4d^2. \end{aligned} \tag{12}$$

These formulas exhibit $P(v, 0) = 0$ and $D_\xi P(v, 0) = 0$ for every $v > 0$. Each term in R_v has order at least three in ξ, because $k - 1 = h^2/(k + 1)$.

4 Uniform positivity and local maximality

Choose a closed interval $I = [v_-, v_+]$ with $0 < v_- < u_0 < v_+ < 1/2$, and set $c_0 = 1 - 4v_+^2 > 0$. The two mixed-variable quadratic blocks in (11), before the factor 2, have matrices

$$\begin{pmatrix} 1 & v \\ v & 1 - 3v^2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -v \\ -v & 1 - 3v^2 \end{pmatrix}.$$

Their determinants are $1 - 4v^2$ and their traces are $2 - 3v^2$. For a positive definite 2 by 2 matrix, its smaller eigenvalue is at least its determinant divided by its trace. Thus the smaller eigenvalue after the factor 2 is at least

$$\frac{2(1 - 4v^2)}{2 - 3v^2} \geq 1 - 4v^2 \geq c_0.$$

The coefficient $16(2 - v^2)$ also exceeds c_0. Hence, with $\rho = \|\xi\|$,

$$Q_v(\xi) \geq c_0 \rho^2 \qquad (v \in I). \tag{13}$$

For completeness, an explicit uniform bound on (12) suffices. If $\rho \leq 1$, then $k \leq \sqrt{2}$, $k - 1 \leq h^2/2$, and

$$|wz + xy|,\ |wy - xz|,\ |wx + yz|,\ |d| \leq r^2/2, \qquad |x^2 + z^2 - y^2 - w^2| \leq r^2.$$

Using $v < 1/2$ and $|a| + |b| \le \sqrt{2}\rho$, the six terms of (12), in their displayed order, are bounded respectively by

$$12\rho^4, \quad 12\sqrt{2}\rho^3, \quad 12\sqrt{2}\rho^3, \quad \sqrt{2}\rho^3, \quad \tfrac{1}{2}\rho^4, \quad \rho^4.$$

Consequently

$$|R_v(\xi)| \le (\tfrac{27}{2} + 25\sqrt{2})\rho^3 < 50\rho^3 \qquad (v \in I,\ \rho \le 1). \tag{14}$$

For $\rho \le \min(1, c_0/100)$, equations (10), (13), and (14) give

$$P(M) \ge c_0\rho^2 - 50\rho^3 \ge \tfrac{1}{2}c_0\rho^2 \ge 0.$$

Continuity of the slice coordinates ensures that every sufficiently nearby conjugate has $v \in I$ and satisfies this bound on ρ. Orthogonal invariance, (6), and (5) therefore give $F(g) \le 1/3$ for every sufficiently nearby left-invariant metric on G_{u_0}.

The estimate is uniform along the flat parameter v; this is why the argument proves a neighborhood maximum despite a degenerate Hessian. The family A_v consists of conjugates of A_{u_0} by $\mathrm{diag}(1, 1, v/u_0, 1)$ for all $v > 0$, and $F(A_v) = 1/3$. In particular the maximum is non-strict. The theorem asserts only the open range $0 < u < 1/2$; neither endpoint is included.

5 The fixed-group obstruction and a larger value

At the candidate, (3) gives, in the order e_0, e_1, e_2, e_3, e_4,

$$\mathrm{Ric} = \mathrm{diag}(-u_0^2/2, 0, 0, u_0^2/2, -u_0^2/2). \tag{15}$$

If $\mathrm{Ric} = cI + D$, the required map $D = \mathrm{Ric} - cI$ is this specific diagonal map. Write its entries $d_0, \dots, d_4$. The derivation identity for $[e_0, e_1] = e_2$ requires $d_2 = d_0 + d_1$. Since $d_1 = d_2 = -c$, it forces $d_0 = 0$ and $c = -u_0^2/2$. Then $d_3 = u_0^2$ and $d_4 = 0$. The nonzero bracket $[e_0, e_4] = u_0 e_3$ instead requires $d_3 = d_0 + d_4 = 0$, a contradiction. This proves failure of the precise equation on $\mathfrak{g}_{u_0}$.

To check nonglobality directly, set

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad T_B = \begin{pmatrix} I_2 & 0 \\ Y & I_2 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

With $H = \mathrm{diag}(1, 1, 1/u_0, 1)$, multiplication gives

$$H A_{u_0} H^{-1} = A_1, \qquad T_B A_1 T_B^{-1} = B.$$

Thus B represents a metric on the same fixed group. Its invariants are $s(B) = 3/2$ and $q(B) = 8$, so

$$F(B) = \frac{(3/2)^2}{(3/2)^2 + 8/4} = \frac{9}{17} > \frac{1}{3}.$$

This proves the last assertion of Theorem 1. Nonglobality for the known family also follows from the prior optimization result [4, Theorem 6.5(iii)]; the displayed comparison is an explicit check independent of that theorem.

6 The Ricci-soliton presentation and disclosure

The obstruction above depends on the transitive group. This distinction can be seen without a classification theorem. Identify the simply connected G_{u_0} with $\mathbb{R} \times \mathbb{R}^4$, with multiplication

$$(t, X)(s, Y) = (t + s, X + e^{tA_{u_0}} Y).$$

Writing $X = (x_1, x_2, x_3, x_4)$, its standard left-invariant metric is

$$dt^2 + \|e^{-tA_{u_0}} dX\|^2 = dx_1^2 + dx_2^2 + dt^2 + (dx_3 - u_0 t\, dx_4)^2 + dx_4^2.$$

The rotation block contributes the Euclidean factor $dx_1^2 + dx_2^2$. The remaining three terms are the standard left-invariant metric on the Heisenberg group H_3 whose orthonormal basis T, Z, W satisfies $[T, W] = u_0 Z$. Hence the Riemannian manifold is the product $\mathbb{R}^2 \times H_3$.

On this product Lie algebra, in the order of the Euclidean basis, T, Z, W, the Ricci operator is $\operatorname{diag}(0, 0, -u_0^2/2, u_0^2/2, -u_0^2/2)$. It equals $cI + D_{\text{prod}}$, where

$$c = -3u_0^2/2, \qquad D_{\text{prod}} = \operatorname{diag}(3u_0^2/2, 3u_0^2/2, u_0^2, 2u_0^2, u_0^2).$$

This map is a derivation: the central entry is the sum of the two noncentral Heisenberg entries, and the Euclidean summand is abelian. The product presentation is therefore a solvsoliton and, consequently, a Ricci soliton: the symmetric derivation integrates to automorphisms, whose pullback metric has derivative $2g(D_{\text{prod}} \cdot, \cdot)$. Thus its Ricci tensor is a scalar multiple of the metric plus a Lie derivative. Theorem 1 does not assert the existence of a local maximum that is not a Ricci soliton.

Status, credit, and AI assistance. This is an unrefereed research note. The known family, value, critical-point restrictions, and nonglobality machinery are credited above; the earlier exact existence announcement is explicit in the abstract and introduction. No new-family claim, firstness claim, established novelty, or current community-open claim is made. The direct proof presented here is intended to be checked from the displayed identities and estimates. AI tools were used extensively in solving the problem, symbolic computation, source review, writing, and adversarial verification. Machine agents are not independent human referees; this work has not received external human peer review. The author retains responsibility for the claims, calculations, citations, and exposition.

References

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[4] J. Lauret and C. E. Will, *The Ricci pinching functional on solvmanifolds*, arXiv:1808.01380v2, 9 May 2019. Journal metadata: *Quarterly Journal of Mathematics* **70** (2019), 1281–1304, doi:10.1093/qmath/haz020. The full text checked here is arXiv v2; no assertion of textual identity with the journal version is made.