Inverse Frobenius in generic multiplicative fields

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Abstract

In every positive-characteristic model of ACFH, inverse Frobenius is a parameter-free definable multiplicative endomorphism outside the integral polynomial ring generated by the generic operator. This gives a negative answer to the unrestricted equality stated in the 2023 Oberwolfach contribution and in arXiv:2212.02115v4, Question 5.8. We include an elementary existential-closure proof of polynomial faithfulness. The characteristic-zero case and the full classification after adjoining inverse Frobenius remain unresolved here.

1 The question and the exact answer

Let T be the theory of algebraically closed fields K with a unary operator θ, where $\theta(0) = 0$ and $\theta|_{K^\times} : K^\times \to K^\times$ is a group endomorphism. The models of ACFH are precisely the existentially closed models of T [2, Corollary 3.15]. Such models exist in every prime characteristic: apply the model-companion embedding [2, Corollary 3.16] to the algebraic closure of $\mathbb{F}_p$ with the identity operator. The embedding preserves characteristic.

Write $E = \mathrm{End}_{\mathrm{def}}(K^\times; \theta)$ for the ring of group endomorphisms definable in the field language expanded by θ. Its addition is pointwise multiplication and its multiplication is composition. Its zero is the constant-one map; its identity is $x \mapsto x$. The integer n denotes $[n](x) = x^n$, with ordinary group inverses when $n < 0$. For $P(X) = \sum_{i=0}^{d} a_i X^i \in \mathbb{Z}[X]$, put

$$P(\theta)(x) = \prod_{i=0}^{d} \theta^i(x)^{a_i}, \qquad x \in K^\times.$$

The image of evaluation is denoted $\mathbb{Z}[\theta]$. These coefficients remain integers even when the field has characteristic p: the operator $[p]$ is Frobenius, rather than the ring zero.

The question whether $E = \mathbb{Z}[\theta]$ is stated without a characteristic restriction in the Oberwolfach contribution [1, pp. 97–98] and in [2, Question 5.8]. We answer that exact unrestricted question negatively.

Theorem 1. *Let (K, θ) be a model of ACFH of characteristic $p > 0$. The map $\rho : K \to K$ defined by $\rho(x)^p = x$ is a parameter-free definable field automorphism commuting with θ. Its restriction to $K^\times$ lies in $E \setminus \mathbb{Z}[\theta]$.*

The inverse-Frobenius map is classical; negative Frobenius powers also appear in [3, Proposition 1.14]. Polynomial faithfulness is already [2, Corollary 5.7]. The counterexample below combines these known ingredients. The next lemma supplies a direct derivation of the latter ingredient, without assuming an endomorphism classification.

2 A checkable proof

We use the elementary extension property of divisible abelian groups: if $H \leq A$ and D is divisible, any homomorphism $H \to D$ extends to A. Here is the needed proof. Take a maximal partial extension $h : B \to D$ by Zorn's lemma. If $a \in A \setminus B$, then $\{k \in \mathbb{Z} : a^k \in B\}$ is either $\{0\}$ or $n\mathbb{Z}$ for $n > 0$. In the first case choose any $d \in D$; in the second choose d with $d^n = h(a^n)$. The formula $h(ba^k) = h(b)d^k$ defines a homomorphism on $B\langle a\rangle$: uniqueness of the expression handles the first case, and the relation subgroup $n\mathbb{Z}$ handles the second. This contradicts maximality, so $B = A$.

Lemma 2. *If (K, θ) is existentially closed in T, evaluation $\mathbb{Z}[X] \to E$ at θ is injective, in every characteristic.*

Proof. Let $Q(X) = \sum_{i=0}^{d} b_i X^i \neq 0$. Choose algebraically independent $t_0, \ldots, t_d$ over K and set $L = K(t_0, \ldots, t_d)$. The subgroup

$$H = K^\times \langle t_0, \ldots, t_{d-1}\rangle \leq L^\times$$

is the direct product of $K^\times$ with the free abelian group on the displayed indeterminates. A nontrivial Laurent-monomial relation over K would contradict algebraic independence. On H define

$$h\left(c \prod_{i<d} t_i^{m_i}\right) = \theta(c) \prod_{i<d} t_{i+1}^{m_i}.$$

For $d = 0$, use $H = K^\times$ and $h = \theta|_{K^\times}$. This is a well-defined homomorphism. Since $L^\times$ is divisible, the extension property gives a homomorphism $\theta_L : L^\times \to L^\times$ extending h. Set $\theta_L(0) = 0$. Thus (L, θ_L) is a T-extension, and $\theta_L^i(t_0) = t_i$ for $0 \leq i \leq d$. Consequently

$$Q(\theta_L)(t_0) = \prod_{i=0}^{d} t_i^{b_i} \neq 1.$$

The transferred existential formula can be written without negative powers or an inverse symbol:

$$\exists x \left(x \neq 0 \ \wedge \prod_{b_i > 0} \theta^i(x)^{b_i} \neq \prod_{b_i < 0} \theta^i(x)^{-b_i} \right).$$

Empty products are 1. It holds in the T-extension, hence in K by existential closedness. Therefore $Q(\theta)$ is not the zero endomorphism. Constants and negative coefficients are included. □

Proof of Theorem 1. Algebraic closedness makes $x \mapsto x^p$ surjective. It is injective because $u^p - v^p = (u - v)^p$. Its inverse ρ has the parameter-free graph $y^p = x$. Unique roots give

$$\rho(xz) = \rho(x)\rho(z), \qquad \rho(x + z) = \rho(x) + \rho(z).$$

Thus ρ is a field automorphism. Multiplicativity of θ gives $\theta(\rho(x))^p = \theta(x)$, so uniqueness also gives $\theta\rho = \rho\theta$, including at zero.

Suppose that $\rho|_{K^\times} = P(\theta)$ for $P \in \mathbb{Z}[X]$. Compose with $[p]$ to obtain

$$(pP - 1)(\theta) = 0_E.$$

Lemma 2 forces $pP - 1 = 0$ in $\mathbb{Z}[X]$. If a_0 is the constant coefficient of P, the constant coefficient of $pP - 1$ is $pa_0 - 1$, nonzero modulo p. This is impossible. □

Corollary 3. *In characteristic $p > 0$, the subring of E generated by $\mathbb{Z}[\theta]$ and ρ is isomorphic to $\mathbb{Z}[1/p][X]$ and strictly contains $\mathbb{Z}[\theta]$.*

Proof. The scalar $[p]$ commutes with every group endomorphism, since $f(x^p) = f(x)^p$. Its inverse ρ is therefore central. Define $P/p^n \mapsto \rho^n \circ P(\theta)$. Clearing common powers of p, which are central units in E, proves well-definedness and compatibility with the ring operations. Composing an element of its kernel with $[p^n]$ gives $P(\theta) = 0_E$, hence $P = 0$ by Lemma 2. Strictness follows from Theorem 1. □

3 Scope, prior work, and verification

Theorem 1 completely refutes the unrestricted equality; Corollary 3 is only an inclusion. Equality with $\mathbb{Z}[1/p][\theta]$ in characteristic p and the characteristic-zero classification are separate questions, unresolved here. Existential closedness is essential: with $\theta = \rho$ on an arbitrary algebraically closed characteristic-p field, $pX - 1$ annihilates θ and $\rho \in \mathbb{Z}[\theta]$. In finite fields inverse Frobenius is an integer power; finite-field experiments therefore cannot establish the theorem's exclusion.

Chini's vector-space classification and subsequent question distinguish the operator reduct from a richer base language that already defines additional endomorphisms [5, Theorem 4.10 and Question 4.11]. This is relevant prior conceptual context. The proposed adaptation to multiplicative fields explicitly involves torsion [4, Remark 2.20]. On $K^\times/\operatorname{Tor}(K^\times)$, inverse Frobenius induces the existing rational scalar $1/p$; the present exclusion concerns the full group and integral coefficients. Those vector-space results do not supply that full-group exclusion.

A bounded primary-source and repository audit found no earlier exact full counterexample; first historical priority is not certified. The final APAL journal body was inaccessible and was not compared with the inspected arXiv v4. Our negative answer targets the explicitly identified OWR and arXiv-v4 formulations; no assertion about the final journal wording or the question's current open status is made. The supporting archive contains the frozen candidate, proof and source audits, priority evidence and clarification, and portable integrity and finite negative controls. Those computations check their stated finite identities and package integrity, rather than formally verify the theorem.

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References

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