

Initial omnific models of discrete initial surreal groups

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Abstract

Every discrete initial ordered additive subgroup of the surreal numbers is order-group-isomorphic to a subgroup of the omnific integers that is initial in the surreal numbers. This holds for sets and proper classes in NBG with global choice and answers the discrete-group question posed by Kaplan and Ehrlich. An explicit operation on ordinal-length sign sequences moves the least exponent to zero without creating a right-simplicity-predecessor obligation. Relabeling the actual normal-form supports and normalizing only the bottom coefficient group then gives the required image. An appendix supplies a detailed sufficiency proof of the initial-group characterization used in the argument, addressing support exhaustion and the legality of limit-cut options.

1 Question, conventions and established inputs

An *initial* subclass of $\mathbf{No}$ contains every proper sign-sequence prefix of each of its members. This is closure under simplicity, not numerical convexity. We write $u \prec_s v$ for a proper prefix, $u \preceq_s v$ when equality is also allowed, $P(v) = \{u : u \prec_s v\}$, and $R_s(v) = \{u \in P(v) : u > v\}$. All orders and group operations are inherited from $\mathbf{No}$. A nontrivial ordered group is *discrete* if it has a least positive element; the zero group is treated separately.

The following is the full additive-group question, with no multiplicative closure hypothesis.

Theorem 1. *Let $G \subseteq \mathbf{No}$ be a discrete initial ordered additive subgroup. There is an order-group isomorphism $T : G \longrightarrow H$, where $H \subseteq \mathbf{Oz}$ is initial in $\mathbf{No}$. The assertion holds when G is a set and when it is a proper class. The zero group also has the stated property.*

The target occurs in Kaplan’s April 2015 honors thesis [1, §7.1, Question 1, p. 47], in his joint-work contribution with Ehrlich [2, Question 1, p. 3356], and in Ehrlich–Kaplan [3, Question 9.1 in the author version]. The numbering in arXiv:1512.04001v1 is Question 2. The isomorphism in Theorem 1 is not required to preserve simplicity.

We work in NBG with global choice, as in [3]. Every surreal has a unique normal form

$$x = \sum_{\mu < \lambda} r_\mu \omega^{y_\mu}, \quad r_\mu \in \mathbb{R} \setminus \{0\}, \quad y_\mu \text{ strictly decreasing,}$$

with set ordinal λ ; zero has empty support. Addition is coefficientwise and order is determined by the leading nonzero coefficient. All coefficient products below are real scalings of monomials. We identify groups with their *canonical* normal-form images, not with an arbitrarily chosen abstract Hahn representation. We use these established results of [3]:

1. The canonical normal-form correspondence is an ordered-group isomorphism (Proposition 3.3).
2. A nonzero initial subgroup of $\mathbb{R}$ is either $2^{-n}\mathbb{Z}$ for $n \in \mathbb{N}_0$, or contains $\mathbb{Z}[1/2]$ (Lemma 5.1).
3. A nonzero additive subgroup A of normal forms is initial precisely when it is truncation closed and cross sectional over an initial exponent class Γ , its pure coefficient groups

$$R_y = \{r \in \mathbb{R} : r\omega^y \in A\}$$

are initial in $\mathbb{R}$, and $\mathbb{Z}[1/2] \subseteq R_y$ whenever $x, y \in \Gamma$ and $y \in R_s(x)$ (Theorem 5.1). Here cross sectional means $\omega^y \in A$ for every $y \in \Gamma$.

4. A normal form lies in $\mathbf{Oz}$ precisely when its exponents are nonnegative and its coefficient at zero, if present, is integral (§6.2).

For truncation-closed groups, every coefficient occurring in an actual series belongs to its pure coefficient group: subtract its two adjacent truncations, using the full series as the later truncation at a last term. The exponent class is therefore exactly the class of occurring exponents when the group is also cross sectional. The appendix checks the sufficiency direction in item 3 without a bounded-support assumption or fractional options outside the group. The underlying standard normal-form and monomial-cut facts remain explicit inputs.

2 An exponent map with the required prefix control

Lemma 2. *If G is nontrivial, discrete and initial, its exponent class Γ has minimum $m = -\alpha$ for a set ordinal α , and $R_m = 2^{-n}\mathbb{Z}$ for some $n \in \mathbb{N}_0$. Its least positive element is $2^{-n}\omega^m$.*

Proof. Let $g > 0$ be least, with leading term $r\omega^m$, $r > 0$. Truncation closure puts this term and the residual tail in G . A nonzero tail has absolute value strictly between zero and g , a contradiction. Thus $g = r\omega^m$. The group R_m has least positive element r , so item 2 gives $R_m = 2^{-n}\mathbb{Z}$ and $r = 2^{-n}$. If $y \in \Gamma$ were below m , cross sectionality would give $0 < \omega^y < g$. Hence $m = \min \Gamma$. A plus sign in m would make the prefix before that sign a member of Γ strictly below m . Thus m is the all-minus sequence $(-)^{\alpha}$, namely $-\alpha$. $\square$

The least-positive-element form is already present in [3, Proposition 5.2]; the negative-ordinal minimum observation also appears in [4, Lemma 3.2]. The proof records their precise role here.

Concatenation of sign sequences is denoted by juxtaposition, and e is the empty sequence. Concatenation is the ordered sum of blocks; no cancellation of ordinal lengths is intended.

Lemma 3. *Let Γ be initial with minimum $a = (-)^{\alpha}$. There is a strictly increasing map $f : \Gamma \rightarrow \mathbf{No}$ such that $\Delta = f[\Gamma]$ is initial, $\min \Delta = 0$, and*

$$f(y) \in R_s(f(x)) \implies y \in R_s(x). \quad (1)$$

Proof. In fact the formula is defined on the full ray $s \geq a$:

$$f(s) = \begin{cases} e, & s = a, \\ (+)s, & s > a \text{ and } a \text{ is not a prefix of } s, \\ (+)at, & s = a(+)t. \end{cases} \quad (2)$$

The cases exhaust the ray because an extension of a above a has a plus as its next sign. Call the second case A and the third B.

Within each case, replacement of a common prefix preserves and reflects order and proper-prefix relations. For a case-A sequence s and a case-B sequence u , either $s = (-)^\beta$ with $\beta < \alpha$, or s first deviates from a by a plus at a position $\beta < \alpha$. In the former case $s > u$ and $s \prec_s u$; its image is likewise a proper prefix above $f(u)$, whose next sign is minus. In the latter case $s > u$, with no prefix comparability, and the same first-deviation comparison holds after the common prepended plus. These alternatives also rule out collisions between the two cases. Thus f is strictly increasing and, away from a , preserves and reflects proper prefixes. Its value at a is zero and all other images are positive.

For every $s > a$ we have the exact identity

$$P(f(s)) = \{e\} \cup \{f(v) : v \in P(s), v \neq a\}. \quad (3)$$

For case A, strip the initial plus. For case B, the proper prefixes of $(+)at$ are exactly e , the sequences $(+)(-)^{\beta}$ with $\beta < \alpha$, and $(+)av$ with v a proper prefix of t . They are respectively $f(a)$, $f((-)^{\beta})$, and $f(a(+)v)$. When $t = e$, the last family is empty: $(+)a$ is the complete image, not an additional proper prefix. Equation (3) and initiality of Γ give initiality of Δ . Finally, a right predecessor in Δ cannot be zero, since zero is its minimum. Prefix reflection away from a and strict monotonicity give (1). $\square$

The argument treats arbitrary ordinal lengths. For instance $1+\alpha = \alpha$ can hold at a limit, but the ordered-block prefix partition in the proof is unchanged. The full-ray inverse is explicit: $h(e) = a$, and, for $z = (+)v$, set $h(z) = a(+)t$ if $v = at$, and $h(z) = v$ otherwise. The two compositions are the identity. For $\alpha = 0$, (2) is the identity on nonnegative surreals. No completion of Γ or transfinite class recursion is used.

3 Transport of the actual subgroup

Proof of Theorem 1. The zero group uses the identity. For a nonzero group take m, n, f from Lemmas 2 and 3. Set $c_m = 2^n$ and $c_y = 1$ for $y \neq m$. Define

$$T \left(\sum_{\mu < \lambda} r_\mu \omega^{y_\mu} \right) = \sum_{\mu < \lambda} c_{y_\mu} r_\mu \omega^{f(y_\mu)}. \quad (4)$$

The support on the right is a set, with the same reverse well-order type as the original. Thus it is a normal form of a surreal. Positive fixed scalings and injective increasing exponent relabeling make T additive, injective and order preserving. Set $H = T[G]$. This is exactly the image of the actual group; no full Hahn group is added.

Its exponent class is Δ , and truncation closure follows because corresponding truncations commute with T . For $y \neq m$, $T(\omega^y) = \omega^{f(y)}$; at the bottom, $T(2^{-n}\omega^m) = 1$. Hence H is cross sectional. A one-term image has a one-term preimage, so its pure coefficient groups are exactly

$$R'_0 = \mathbb{Z}, \quad R'_{f(y)} = R_y \quad (y \neq m). \quad (5)$$

They are initial. If $f(y) \in R_s(f(x))$, then $y \in R_s(x)$ by (1), and $y \neq m$. Thus $\mathbb{Z}[1/2] \subseteq R_y = R'_{f(y)}$. Every hypothesis of the initial-group characterization holds, and H is initial in $\mathbf{No}$. Its exponents are nonnegative. By truncation closure, every occurring zero coefficient belongs to $R'_0 = \mathbb{Z}$. The omnific criterion therefore gives $H \subseteq \mathbf{Oz}$.

For a proper class G , the parameters α, n are still sets. The explicit formula (2) defines a class function, and class replacement sends each individual support to a set. Equation (4) defines T and its image class by elementary comprehension. Every group calculation uses finitely many set supports. The characterization applies to classes as justified in the appendix. There is no proper-class support, class-sized sum, or stronger class-recursion assumption. $\square$

4 Attribution, verification and limits

Exponent relabeling and positive coefficient scaling are established Hahn-group mechanisms; an explicit later account is [5, Proposition 3.9 and Construction 3.10]. General order/simplicity substructures and partition constructions [6] do not by themselves guarantee the additive compatibility and initial-image property needed here. The specific bridge proved above is the sign operation (2), its right-prefix control, and its use in the actual subgroup. No claim of first discovery of the general transport machinery is made.

The source package records extensive bounded primary-source priority searches by the author and three internal AI review approaches. No earlier explicitly recorded full solution was located in the inspected accessible material. This is not a certificate of novelty, firstness, causal independence or continued global openness. Some publisher versions were not authenticated against the author/arXiv versions. Material unread fulltext leads include Ehrlich's 2024 handbook chapter [7], the full 2001 predecessor and other access-limited works specified in the priority ledger. Only the 2024 chapter's primary abstract and references were read; no absence claim is made about its unread contents.

The portable verification program checks finite sign trees, prefix identities, right-predecessor implications, rational finite-support additivity and bottom normalization. Ordinary and optimized runs compare the full expected output. Four altered constructions fail computed properties in each mode. These finite diagnostics do not establish the ordinal-length or proper-class theorem, which is proved above. Historical reports mention unavailable earlier programs; those are not represented as reproduced by this package.

AI tools were used extensively in developing, writing, researching, reproducing and adversarially checking this work. Three initial full mathematical approaches and a further focused dependency reconstruction were internally reviewed; exact exposure histories and source-reading limits are recorded in the package. This preprint is unrefereed. Internal AI reviews are not external human peer review, journal acceptance or formal proof-assistant certification.

A A detailed proof of the initial-group sufficiency criterion

This appendix proves the sufficiency assertion used above. The characterization is credited to [3, Theorem 5.1], not claimed as new. In the inspected author proof two literal steps require care: the union of bounded exponent stages need not exhaust cofinal supports, and dyadic perturbations of a series need not belong to the given coefficient groups. The proof below avoids both steps. This concerns proof presentation, not a counterexample to the theorem.

Proposition 4. *Let A be a truncation-closed, cross-sectional additive subgroup of normal forms over an initial exponent class Γ . Suppose every R_y is initial in $\mathbb{R}$ and $\mathbb{Z}[1/2] \subseteq R_y$ for $y \in R_s(x)$, $x, y \in \Gamma$. Then the corresponding subgroup of $\mathbf{No}$ is initial.*

We use standard recursive addition and negation, the simplicity theorem, normal-form unique-

ness and comparison, the monomial cut

$$\omega^y = \{ 0, k\omega^{y^L} \mid 2^{-k}\omega^{y^R} \}, \quad (6)$$

where $k = 1, 2, \dots$, $y^L \in P(y)$ with $y^L < y$, and $y^R \in R_s(y)$, the pure-multiple cuts of [3, Lemma 3.1(ii)–(iii)], and the usual infinite limit-series cut of [3, Proposition 3.2(iii)]. These normal-form facts, and the real-coefficient classification, are imports; this appendix does not rederive all foundational surreal theory.

A.1 Initial sums and adjoining a cut

If $q = \{L \mid R\}$ and $z \prec_s q$, then z is a prefix of or equal to an option on its side. Indeed, if $z < q$ and every $l \in L$ were below z , then $L < z < R$, contrary to simplicity of q . Thus $z \leq l < q$ for some $l \in L$. A sign sequence lying between a sequence and one of its extensions must extend the shorter sequence, by the first differing sign comparison. Consequently $z \preceq_s l$. For $z > q$ use a member $r \in R$ with $q < r \leq z$. In particular, if all options belong to an initial set M , all proper prefixes of q belong to M .

If S, U are initial sets, then $S + U$ is initial. For each pair (s, u) , induct on the natural (Hessenberg) sum of their ordinal birthdays. Every recursive addition option $s^L + u$, $s + u^L$, $s^R + u$ or $s + u^R$ has a smaller operand measure. It belongs to $S + U$ and, by induction, so do all its prefixes. The preceding option-covering argument for the defining cut of $s + u$ now puts all prefixes of $s + u$ in $S + U$. The induction is on operand birthdays, not on the birthday of the sum.

Negation reverses all signs and preserves prefixes. Hence the additive subgroup generated by any initial set containing zero is initial: take finite sums of its union with its negative, and then their union. It follows that, for an initial set-sized subgroup M and a surreal b simplest between sets of options from M , the group $M + \mathbb{Z}b$ is initial. This directly supplies the additive adjunction needed here; no general module-adjunction theorem is assumed. If desired, when $b \notin M$, its small options extend to the partition $M_{<b}, M_{>b}$: b realizes it, and any realization of the full partition realizes the small cut, so b is still simplest.

A.2 Finite-support elements first

First suppose $\Gamma = E$ is a set. Then A is a set, since all its series are encoded in the set of real-coefficient functions on subsets of E . Well order E by nondecreasing exponent birthday, breaking ties by choice. All proper prefixes of an exponent precede it. Starting with $\{0\}$, build initial subgroups by adjoining actual pure terms.

At exponent y , first adjoin ω^y if necessary. All options in (6) are in the previously generated group: earlier exponent monomials give the left options, and $\mathbb{Z}[1/2] \subseteq R_{y^R}$ gives the right ones. The additive adjunction proves initiality and supplies all integer multiples. Next adjoin the noninteger $r\omega^y$, $r \in R_y$, in nondecreasing real birthday. Initiality of R_y puts its real predecessors in R_y ; their multiples have already appeared (integer multiples are already present). For nondyadic r , or noninteger dyadic r with no left prefix of y , the pure cut uses just $r^L\omega^y$ and $r^R\omega^y$. In the other noninteger dyadic case its options are

$$r^L\omega^y + k\omega^{y^L}, \quad r^R\omega^y - k\omega^{y^L},$$

with $k \in \mathbb{N}_0$ and y^L a left prefix of y . They too are in the preceding group. The pure-multiple cut and adjunction apply in every case. At limits take unions. The resulting initial subgroup F is exactly the additive subgroup generated by all pure terms of A . It contains every finite-support member of A , and only elements of A .

A.3 Legal cuts for all actual limit-length supports

Let $x = \sum_{\mu < \lambda} r_\mu \omega^{y_\mu} \in A$, with λ an infinite limit ordinal, and set $s_\eta = \sum_{\mu < \eta} r_\mu \omega^{y_\mu}$. For $\mu < \lambda$ put

$$l_\mu = s_{\mu+1} - \omega^{y_\mu}, \quad u_\mu = s_{\mu+1} + \omega^{y_\mu}. \quad (7)$$

These are in A , by truncation closure and cross sectionality, and $l_\mu < x < u_\mu$ by leading-coefficient comparison. They use integer, not dyadic, perturbations.

They define x as their simplest separator. Indeed, for any $\mu < \lambda$ and $k \geq 1$, choose ν with $\mu < \nu < \lambda$. Normal-form comparison gives

$$s_{\mu+1} - 2^{-k} \omega^{y_\mu} < l_\nu < x < u_\nu < s_{\mu+1} + 2^{-k} \omega^{y_\mu}. \quad (8)$$

Thus any separator of all the unit options is a separator of the standard limit-series cut, whose simplest element is x . As x itself separates the unit options, it is their simplest separator. We do not assert that every option is itself a prefix of x .

Well order all actual members of A of infinite limit support length, first by that length and then arbitrarily within each length. Start from F and adjoin them successively. At the stage for x of length λ , every member of A of smaller support length is already present. To see this, every ordinal ρ is uniquely $\delta + j$, where $j < \omega$ and δ is zero or a limit ordinal. A series of length ρ is its δ -prefix, if nonempty, plus j pure terms. Its infinite-limit prefix was adjoined at a smaller length, or it is already finite. Each option in (7) has length at most $\mu + 1 < \lambda$, so belongs to the previous subgroup. Adjunction applies. At limit stages take unions. The same $\delta + j$ decomposition proves that the resulting initial group is all of A . This includes supports cofinal in the entire exponent enumeration; no bounded-stage union is used.

A.4 Class exponents via actual set restrictions

For an arbitrary exponent class and $x \in A$, its support S is a set. Let

$$E = \bigcup_{y \in S} (\{y\} \cup P(y)), \quad A_E = \{b \in A : \text{supp}(b) \subseteq E\}.$$

For $x = 0$, one may take $E = \emptyset$. Replacement and union make E a set, and initiality of Γ makes E initial and contained in Γ . The group A_E is a set, as a subclass of the set of all real-coefficient series over E . It inherits truncation closure, cross sectionality, the *same* pure coefficient groups at each $y \in E$, and the right-prefix condition. The set proof makes A_E initial. Since $x \in A_E$, every prefix of x is in $A_E \subseteq A$. This proves Proposition 4 for classes. There is no class-sized cut or class enumeration.

For completeness, necessity uses the normal-form predecessor properties of [3, Proposition 3.2(i)–(ii), Lemma 3.1(i),(iv)]: initiality gives truncation closure and initiality of the occurring exponent class; subtraction of adjacent truncations isolates pure terms, whose simpler leaders are present. It makes each R_y initial and puts $2^{-k} \omega^y$ in the group whenever $y \in R_s(x)$ and ω^x is present. Additive closure gives $\mathbb{Z}[1/2] \subseteq R_y$. These are the same explicit foundational imports as above.

References

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